Tracking a target in spherical coordinates

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TRACKING A TARGET IN SPHERICAL COORDINATES

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Tracking a Target in Spherical Coordinates

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BRIEF

Most tracking devices resolve the target's position into spherical coordinates. The first and second derivatives with respect to time of these coordinates are responsible for practical limitations.

The only assumption made is that targets move in straight level paths. In general, targets may approach from any azimuth angle hence a circular symmetry about a vertical axis.

Azimuth velocities and acceleration are entirely independent of target height or elevation angle and hence have cylindrical "forbidden zones" associated with them.

In general elevation velocities and acceleration will be smaller than corresponding azimuth quantities and if identical controls are provided in elevation and azimuth, the elevation forbidden zones will be entirely within the azimuth forbidden zones.

The various relations developed are illustrated graphically and useful design sheets are included showing the relation between target velocity, minimum range and angular velocity and acceleration.

The interval of angular velocities required to track targets between given maximum and minimum ground ranges is estimated.

The effect of a diving target is estimated.

TRACKING A TARGET IN SPHERICAL COORDINATES

The majority of tracking devices resolve the radius vector, representing the position of the target relative to the tracking device, into spherical coordinates R, θ, and ϕ as illustrated in Figure 1. These quantities are:

R - Slant range or merely range.

θ - Azimuth or bearing angle.

ϕ - Elevation angle or angle of sight.

It is in the time variation of these coordinates namely their velocities and accelerations that we are interested.

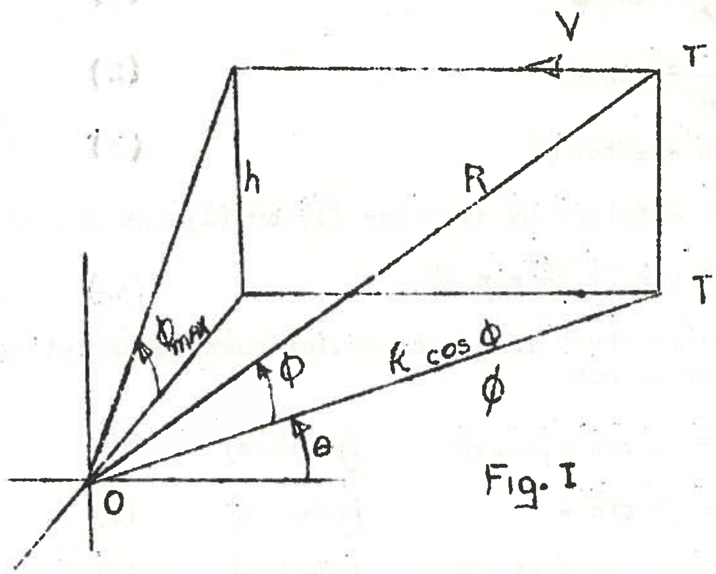


Fig. I

It will be very desirable to simplify the analysis by first assuming that aircraft fly straight, horizontal courses at uniform speed. This is a good first approximation to the actual conditions, and later the effect of diving targets will be estimated. It will be noted that the direction of zero azimuth angle (θ) is taken parallel to the target's course and in the opposite sense. True azimuth (with respect to North) can be obtained by adding θ_o, the angle between our reference direction and North. Since in general, targets can fly courses having any value of θ_o between 0 and 2π, this gives rise to circular symmetry about the vertical axis $\phi = \frac{\pi}{2}$

The path of a target flying a straight level course may be described in terms of the following parameters:-

d - minimum ground range.
h - altitude or height.
ϕ max = $\tan^{-1}\left(\frac{h}{d}\right)$ - maximum elevation angle (for a given course)
V - speed of target.

(all quantities are expressed in units of yards, radians and seconds - except in the case of graphs where degrees, miles per hour and revolution per minute may be more convenient)

The expressions for the instantaneous values of R, θ and φ in terms of the course parameters and time are complicated and need not be used as a starting point in finding the first and second derivatives of R, θ and φ. Instead we proceed to find velocities and accelerations as follows:

From Figure 1 we can write

$$\frac{h}{R\cos\varphi} = \tan\varphi \qquad (1)$$

$$\frac{d}{R\cos\varphi} = \sin\theta \qquad (2)$$

$$\frac{h}{d}\sin\theta = \tan\varphi \qquad (3)$$

Equation (3) is obtained by dividing (1) by (2) and can be rewritten

$$\varphi = \tan^{-1}\left(\frac{h}{d}\sin\theta\right) \qquad (3a)$$

Resolving the velocity V into rectangular components lying in the R, θ and φ directions we get

$$V_R = V\cos\theta\cos\varphi \qquad [\text{yds/sec}] \qquad (4)$$

$$V_\theta = V\sin\theta \qquad [\text{yds/sec}] \qquad (5)$$

$$V_\varphi = V\cos\theta\sin\varphi \qquad [\text{yds/sec}] \qquad (6)$$

Expression (4) gives the range velocity directly. The azimuth and elevation angular velocities are found by dividing (5) and (6) respectively by their proper radii R cos φ, and R thus we get

$$\frac{dR}{dt} = -V\cos\theta\cos\varphi \qquad [\text{yds/sec}] \qquad (7)$$

$$\frac{d\theta}{dt} = \frac{V\sin\theta}{R\cos\varphi} \qquad [\text{rad/sec}] \qquad (8)$$

$$\frac{d\varphi}{dt} = \frac{V\cos\theta\sin\varphi}{R} \qquad [\text{rad/sec}] \qquad (9)$$

The relations (1) to (9) are illustrated in fig. 2 and 3.

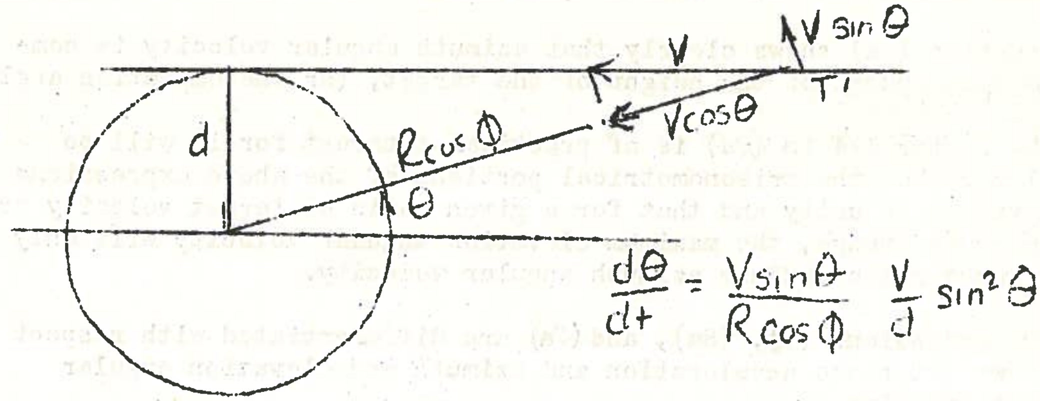


Fig. 2

Fig. 2 is a plan view and shows the target velocity V broken into two horizontal components V cos θ, and V sin θ. The latter component, V sin θ, is in the "θ" direction.

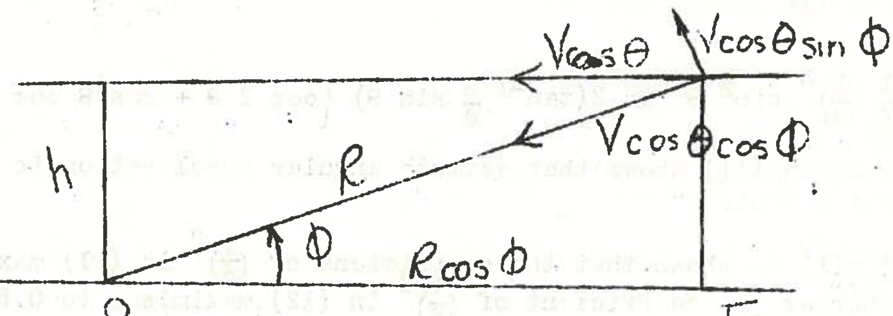


FIG. 3

Fig. 3 is a vertical section taken so as to include the radius vector OT, and shows the component of velocity, V cos θ, in this plane, further broken up; V cos θ cos φ in the R direction and V cos θ sin φ in the φ direction.

Expressions (8) and (9) are of particular interest, being the azimuth and elevation angular velocities respectively. However in the above form they are variables of all three spherical coordinates and we must make use of expressions (1), (2) and (3) to reduce (8) and (9) to functions of a single variable. The azimuth angle θ was chosen as the variable, and (8) and (9) in their final form are

$$\frac{d\theta}{dt} = \frac{V}{d} \sin^2 \theta \qquad [\text{rad/sec}] \qquad (8a)$$

$$\frac{d\varphi}{dt} = \frac{1}{4} \frac{V}{d} \sin 2\theta \sin 2(\tan^{-1} \frac{h}{d} \sin \theta) \qquad (9a)$$

Equation (8a) shows clearly that azimuth angular velocity is completely independent of the height of the target, (or the elevation angle).

The factor 1/4 in (9a) is of practical interest for it will be shown later that the trigonometrical portions of the above expressions both maximise to unity and that for a given ratio of target velocity to minimum ground range, the maximum elevation angular velocity will only be one quarter the maximum azimuth angular velocity.

If expressions (7), (8a), and (9a) are differentiated with respect to time we get range acceleration and azimuth and elevation angular accelerations viz:

$$\frac{d^2R}{dt^2} = -\frac{V^2}{d} \sin\theta \cos\varphi \, (\sin^2\theta + \cos^2\theta \sin^2\varphi) \qquad [\text{yds/sec}^2] \qquad (10)$$

$$\frac{d^2\theta}{dt^2} = 2\left(\frac{V}{d}\right)^2 \sin^3\theta \cos\theta \qquad [\text{rad/sec}^2] \qquad (11)$$

$$\frac{d^2\varphi}{dt^2} = \frac{1}{2}\left(\frac{V}{d}\right)^2 \sin^2\theta \sin 2(\tan^{-1}\frac{h}{d}\sin\theta) \left\{\cos 2\theta + \cos^2\theta \cos 2(\tan^{-1}\frac{h}{d}\sin\theta)\right\} \qquad (12)$$

Equation (11) shows that azimuth angular acceleration is independent of target height.

It will be shown that the coefficient of $(\frac{V}{d})^2$ in (11) maximises to 0.65, whereas the coefficient of $(\frac{V}{d})^2$ in (12) maximises to 0.50 so that the maximum elevation angular acceleration will be about 77% of the maximum azimuth angular acceleration for a given ratio of target velocity to minimum ground range.

This completes the number of expressions to be derived and the more interesting ones are illustrated in the form of graphs.

For complete derivation of the various expressions above, see Appendix.

GRAPH "1" - Expression (3) may be rewritten

$$\varphi = \tan^{-1}\left(\frac{h}{d}\sin\theta\right) \qquad (3a)$$

and the ratio $\left(\frac{h}{d}\right) = \tan\varphi$ max .
may be taken as a parameter. Curve sheet 1 shows a family of such curves for values of $\left(\frac{h}{d}\right)$ ranging from $\frac{1}{2}$ to 10. This serves to give an idea of how φ and θ will vary. This set of curves will be used later to show contours of equal elevation angular velocities and accelerations.

It will be noted in passing, that for small angles, the slope of each curve approaches the value of $\left(\frac{h}{d}\right)$ for that curve, as would be expected from the simplified form of (3a) for small angles, i.e.

$$\varphi \doteq \frac{h}{d}\theta \qquad [\text{radians}] \qquad (3b)$$

At $\theta = \frac{\pi}{2} = 90^{\circ}$ all curves have zero slope. This last fact will show up as a zero elevation angular velocity at $\theta = \frac{\pi}{2} = 90^{\circ}$ (see curve sheet 4).

GRAPH "2"

Expression (8a) is plotted for various values of "d" and one speed of 100 m.p.h. and shows clearly that there is a maximum azimuth velocity at $\theta = \frac{\pi}{2} = 90^\circ$. For the purpose of assigning values to the top speed delivered from the azimuth driving mechanism expression (8a) is plotted in another form in GRAPH 3.

GRAPH "3"

Setting $\theta = \frac{\pi}{2}$ we get the maximum of $\sin^2\theta$ which equals unity and we are left with

$$\frac{d\theta}{dt} = \frac{V}{d} \qquad [rad/sec] \quad . \ . \ .$$

which gives a family of straight lines of slope (-1) when plotted on loglog graph paper.

This is a useful design sheet.

GRAPH "4"

Expression (11) is plotted for various values of "d" and one speed of 100 m.p.h. and illustrates the maximum of acceleration in azimuth at $\theta = \frac{\pi}{3} = 60^{\circ}$ and the zero at $\theta = \frac{\pi}{2} = 90^{\circ}$, (where the slope of the velocity curve was zero). The maximum at $\theta = \frac{\pi}{3}$ may be confirmed by setting to zero the derivative of expression (11) with respect to θ and solving for θ.

GRAPH "5"

When the value $\theta = \frac{\pi}{3}$ is inserted into expression (11) the coefficient of $(\frac{V}{d})^2$ becomes 0.65 and the expression for maximum azimuth angular acceleration is

$$\frac{d^2\theta}{dt^2} = 0.65(\frac{V}{d})^2 \qquad [\mathrm{rad/sec^2}]$$

which gives a family of straight lines of slope -2 when plotted on loglog paper.

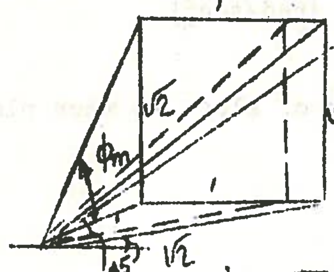
This is a useful design sheet.

GRAPH "6"

Of the complete expression (9a) for elevation angular velocity, only the trigonometric portion is plotted here. Various values of the parameter $\frac{h}{d} = \tan \varphi$ max are used, and the resulting family of curves have as their envelope $f(\theta) = \sin 2\theta$. The maximum of this envelope obviously occurs at $\theta = \frac{\pi}{4} = 45°$. By rewriting expression (9a) as a variable of both θ and φ we get

$$\frac{d\varphi}{dt} = \frac{1}{4} \left(\frac{V}{d}\right) \sin 2\,\theta \sin 2\,\varphi \qquad (9b)$$

It can then be seen by inspection that the path which gives the greatest elevation velocity for a given ratio $\left(\frac{V}{d}\right)$ is the one where θ and φ are simultaneously $\frac{\pi}{4}$ or 45°.



The value of $\frac{h}{d}$ for this track is easily found with the aid of the diagram, or from expression 3.

$$\frac{h}{d} = \frac{\tan \varphi}{\sin \theta} = \frac{1}{0.707} = \sqrt{2}$$

The curve for $\frac{h}{d} = \sqrt{2}$ is shown dotted, and the angle $\varphi_m = \tan^{-1} \sqrt{2}$ is 54.6°

GRAPH "7"

Taking the maximum elevation angular velocity

$$\frac{d\varphi}{dt} = \frac{1}{4}\left(\frac{V}{d}\right) \qquad [\text{rad/sec}] \qquad \text{as } 100\%$$

we can map contours of equal velocity on a chart of φ vs θ. It turns out that the contours are circles having

the point $\theta = \frac{\pi}{4}$, $\varphi = \frac{\pi}{4}$ and radii,

$$r = r_{max} \frac{\cos^{-1}\left(1 - \frac{2P}{100}\right)}{\pi}$$

where P is the value of angular velocity expressed in percent of the maximum.

By following the path of any course across the diagram we can see what velocities are going to be encountered, and several courses can be compared at a glance. Likewise an idea can be had of what the proportion of courses through space will involve high rates of tracking in elevation.

The dotted curve crossing all the course paths is the locus of "maximum for a given course". It is interesting to note in passing that this curve is merely the $\frac{h}{d} = \sqrt{2}$ curve of the φ vs θ family rotated 90° on the paper.

i.e. $\theta = \tan^{-1}(\sqrt{2}\cos\varphi)$

GRAPH "8"

Elevation angular acceleration is given by expression (12). Here again only the trigonometric portion is plotted, for various values of $(\frac{h}{d}) = \tan\varphi max.$ It will be seen that for small angles the acceleration is positive, i.e. tending to ~~the~~ move the line of sight upwards, then depending upon the path, a maximum velocity is reached and the acceleration is reduced to zero and reverses. The negative maxima for all paths occur at $\theta = \frac{\pi}{2} = 90°$ and the path "$\frac{h}{d} = 1$" yields the all time high in elevation acceleration under this combination of conditions

For $\theta = \frac{\pi}{2}$ and $\varphi = \varphi max = \frac{\pi}{4}$ the trigonometric expression reduces to unity and the value of angular acceleration is

$$\frac{d^2\varphi}{dt^2} = 0.5 \left(\frac{V}{d}\right)^2 \qquad \left[rad/sec^2\right] \qquad (12a)$$

as compared with

$$\frac{d^2\theta}{dt^2} = 0.65 \left(\frac{V}{d}\right)^2 \qquad \left[rad/sec^2\right] \qquad (11a)$$

for azimuth.

GRAPH "9"

Taking the maximum elevation angular acceleration

$$\frac{d^2\varphi}{dt^2} = 0.5\left(\frac{V}{d}\right)^2 \quad \text{as } 100\%$$

we can map contours of equal acceleration, again on the φ vs θ chart. The resulting contours do not exhibit the simplicity of the velocity contours. It will be noted however that the curve of maximum velocity map shows up as a zero of acceleration. In all the region to the left of this line there is positive acceleration, to the right, negative acceleration.

GRAPH "10"

Returning to the expression used in Graph 3.

$$\frac{d\theta}{dt} = \frac{V}{d} \qquad \text{or} \qquad d = \frac{V}{\frac{d\theta}{dt}}$$

we see that for a given target velocity "V" and a given installed maximum azimuth velocity, there is a corresponding minimum value of "d" smaller than which the minimum ground range must not go if tracking is to be continuous. Since no restrictions on "θ_0" or height "h" are involved, the resulting "forbidden" volume, through which the target's path must not go, is a circular cylinder of

radius [illegible] $= \dfrac{V}{\left.\frac{d\theta}{dt}\right|_{max}}$ and of infinite height. This is the condition imposed by a limited azimuth angular velocity.

There is a corresponding condition imposed by a limited elevation angular velocity, only in this case the shape of the forbidden volume is not so simple. An idea of its shape may be obtained from the peak values of the curves of $\frac{d\varphi}{dt}$ in Graph 6. Rewriting the expression for $\frac{d\varphi}{dt}$, making "d" a function of $(\frac{h}{d})$ as well as of (V) $\left.\frac{d\varphi}{dt}\right|_{max}$ and (θ) we get

$$d_{min} = \frac{1}{4} \frac{V}{\left.\frac{d\varphi}{dt}\right|_{max}} \left(f\left(\frac{h}{d},\theta\right)\right)_{peak}$$

and the corresponding value of (h) is

$$h = \left(\frac{h}{d}\right) d_{min} = \frac{h}{d} \cdot \frac{1}{4} \frac{V}{\left.\frac{d\varphi}{dt}\right|_{max}} f\left(\frac{h}{d},\theta\right)_{peak}$$

Thus we can make a table of (d) and (h) expressed in units of $\dfrac{1}{4} \dfrac{V}{d\varphi/dt|_{max}}$ for various values of $\frac{h}{d}$

$\frac{h}{d}$	d	h
1/2	0.63	0.31
1	0.95	0.95
$\sqrt{2}$	1.00	1.41
2	0.95	1.90
3	0.81	2.43
10	0.36	3.60

and we can plot a curve of (h) vs. (d) showing the cross section of the volume of revolution generated by this curve. This is the forbidden region associated with a limited elevation angular velocity.

Since $f(\frac{h}{d}, \theta)$ is never greater than unity, it will be seen that for a given target velocity V and equal maximum azimuth and elevation velocities, the elevation forbidden region lies entirely within the azimuth forbidden cylinder. If the given maximum elevation velocity is retained, but the azimuth velocity extended, the azimuth forbidden cylinder will shrink in inverse ratio until at 4 to 1 azimuth to elevation velocity ratio, the surfaces become tangent.

GRAPH "11"

It is sometimes desirable to estimate the ratio of maximum to minimum angular velocities involved in the working range of a tracking device. In the azimuth case we have

$$\frac{d\theta}{dt} = \frac{V}{d}\sin^2\theta \doteq \frac{Vd}{R^2}$$

thus for a given (d) the minimum azimuth velocity will occur at maximum range R_{max} and

$$\left.\frac{d\theta}{dt}\right|_{min} = \frac{Vd}{R_{max}^2}$$

likewise for a given (d) the maximum azimuth velocity will be

$$\left.\frac{d\theta}{dt}\right|_{max} = \frac{V}{d}$$

hence the ratio

$$S = \frac{\left.\frac{d\theta}{dt}\right|_{max}}{\left.\frac{d\theta}{dt}\right|_{min}} = \frac{\frac{V}{d}}{\frac{Vd}{(R_{max})^2}} \doteq \left(\frac{R_{max}}{d}\right)^2$$

It will be noted that this ratio is independent of the actual velocity of the target and depends only on the minimum and maximum ranges.

On loglog paper the curves of (S) vs (d) are straight lines of slope (-2). Several curves are shown for the various values of R_{max} written on the curves.

EFFECT OF DIVING TARGETS

If the target has a vertical component of celocity V_v the corresponding elevation angular velocity can be shown to be

$$\left.\frac{d\varphi}{dt}\right|_v = \frac{V_v}{d} \sin\theta \cos^2(\tan^{-1}\frac{h}{d}\sin\theta)$$

It is reasonable to believe that unless the target is fairly high up, he will not be diving steeply. For ratios of $(\frac{h}{d})$ greater than two the trigonometric function will not exceed 0.25 and thus

$\left.\frac{d\varphi}{dt}\right|_v < 0.25\,\frac{V_v}{d}$. If in addition the vertical and horizontal velocities are made equal (i.e. target diving at 45°), the total combined elevation angular velocity may be as great as

$$\frac{d\varphi}{dt} = \doteq 0.25\,\frac{V_H}{d} + 0.25\,\frac{V_v}{d} = 0.5\,\frac{V_H}{d} \text{ or } 0.$$

In the rather extreme case the elevation angular velocity can be double the maximum elevation angular velocity incurred with a level flying target of velocity V_H. Hence an azimuth to elevation velocity ratio of only 2 should be allowed.

Because of the increased sensitivity of control obtained by limiting the maximum velocities, it will be desirable to make this azimuth to elevation velocity ratio as close to 4 as is safe depending upon the class of service the tracking device is to perform.

APPENDIX

Having established equations (7), (8) and (9)

$$\frac{dR}{dt} = -V\cos\theta\cos\varphi \qquad [\text{yds/sec}] \qquad (7)$$

$$\frac{d\theta}{dt} = \frac{V\sin\theta}{R\cos\varphi} \qquad [\text{rad/sec}] \qquad (8)$$

$$\frac{d\varphi}{dt} = \frac{V\cos\theta\sin\varphi}{R} \qquad [\text{rad/sec}] \qquad (9)$$

we must make use of (1), (2) and (3)

$$\frac{h}{R\cos\varphi} = \tan\varphi \qquad (1)$$

$$\frac{d}{R\cos\varphi} = \sin\theta \qquad (2)$$

$$\frac{h}{d}\sin\theta = \tan\varphi \qquad (3)$$

if we want to reduce (8) or (9) to functions of a single variable.

From (2) we see that

$$\frac{1}{R\cos\varphi} = \frac{\sin\theta}{d} \qquad (2.1)$$

which when substituted in (8) gives directly

$$\frac{d\theta}{dt} = \frac{V\sin^2\theta}{d} \qquad [\text{rad/sec}] \qquad (8a)$$

Again from (2)

$$\frac{1}{R} = \frac{\sin\theta\cos\varphi}{d} \qquad (2.2)$$

and this together with (9) gives

$$\frac{d\varphi}{dt} = \frac{V}{d}\sin\theta\cos\theta\sin\varphi\cos\varphi \qquad (9.1)$$

If we make use of the trigonometrical transformation

$$\sin\alpha\cos\alpha = \tfrac{1}{2}\sin 2\alpha$$

we arrive at

$$\frac{d\varphi}{dt} = \frac{1}{4}\frac{V}{d}\sin 2\theta\sin 2\varphi \qquad (9.2)$$

On substituting the value of ϕ from Equation 3, we get finally

$$\frac{d\phi}{dt} = \frac{1}{4}\frac{V}{d} \sin 2\theta \sin 2(\tan^{-1} \frac{h}{d} \sin \theta) \qquad (9a)$$

Differentiating (7) with respect to time

$$\frac{d^2R}{dt^2} = - V \left(\sin \theta \cos\phi \frac{d\theta}{dt} + \cos \theta \sin\phi \frac{d\phi}{dt}\right)$$

Substituting values of $\frac{d\theta}{dt}$ (from 8a) and $\frac{d\phi}{dt}$ (from 9.1)

$$\frac{d^2R}{dt^2} = - V\left(\frac{V}{d} \sin^3\theta \cos\phi + \frac{V}{d} \sin \theta \cos^2\theta \sin^2 \phi \cos\phi\right)$$

$$= - \frac{V^2}{d} \sin \theta \cos\phi(\sin^2 \theta + \cos^2 \theta \sin^2\phi) \qquad [\text{yds/sec}^2] \quad (10)$$

Differentiating (8a) with respect to time

$$\frac{d^2\theta}{dt^2} = \frac{2V}{d} \sin \theta \cos \theta \frac{d\theta}{dt}$$

and reinserting $\frac{d\theta}{dt}$ from (8a)

$$\frac{d^2\theta}{dt^2} = 2\left(\frac{V}{d}\right)^2 \sin^3 \theta \cos \theta \qquad [\text{rad/sec}^2] \quad (11)$$

Differentiating (9.2) with respect to time

$$\frac{d^2\phi}{dt^2} = \frac{V}{4d}\left(2 \cos 2 \theta \frac{d\theta}{dt} \sin 2\phi + 2 \sin 2 \theta \cos 2\phi \frac{d\phi}{dt}\right)$$

Substituting values of $\frac{d\theta}{dt}$ (from 8a) and $\frac{d\phi}{dt}$ (from 9.1)

$$\frac{d^2\phi}{dt^2} = \frac{V}{4d}\left(2 \cos 2 \theta \sin 2\phi \frac{V}{d} \sin^2\theta + 2 \sin 2 \theta \cos 2\phi \frac{V}{d} \sin \theta \cos \theta \sin\phi \cos\phi\right)$$

$$= \frac{1}{2}\left(\frac{V}{d}\right)^2 (\sin^2 \theta \cos 2 \theta \sin 2\phi + \sin^2 \theta \cos^2 \theta \sin 2 \phi \cos 2\phi)$$

$$= \frac{1}{2}\left(\frac{V}{d}\right)^2 \sin^2 \theta \sin 2\phi (\cos 2 \theta + \cos^2 \theta \cos 2\phi)$$

on substituting for "ϕ" from (3) we arrive at

$$\frac{d^2\phi}{dt^2} = \frac{1}{2}\left(\frac{V}{d}\right)^2 \sin^2 \theta \sin 2(\tan^{-1} \frac{h}{d} \sin \theta) \left\{\cos 2 \theta + \cos^2 \theta \cos 2\left(\tan^{-1} \frac{h}{d} \sin \theta\right)\right\}$$

$$[\text{rad/sec}^2] \quad (12)$$

APPENDIX

this is the form that was used in calculating the curves. It can be expressed in equivalent form

$$\frac{d^2\phi}{dt^2} = \frac{1}{2}\left(\frac{V}{d}\right)^2 \frac{1 - \cos 2\theta}{2} \sin 2\varphi \left\{\cos 2\theta + \frac{1 + \cos 2\theta}{2} \cos 2\varphi\right\}$$

which would make for easier tabulation.

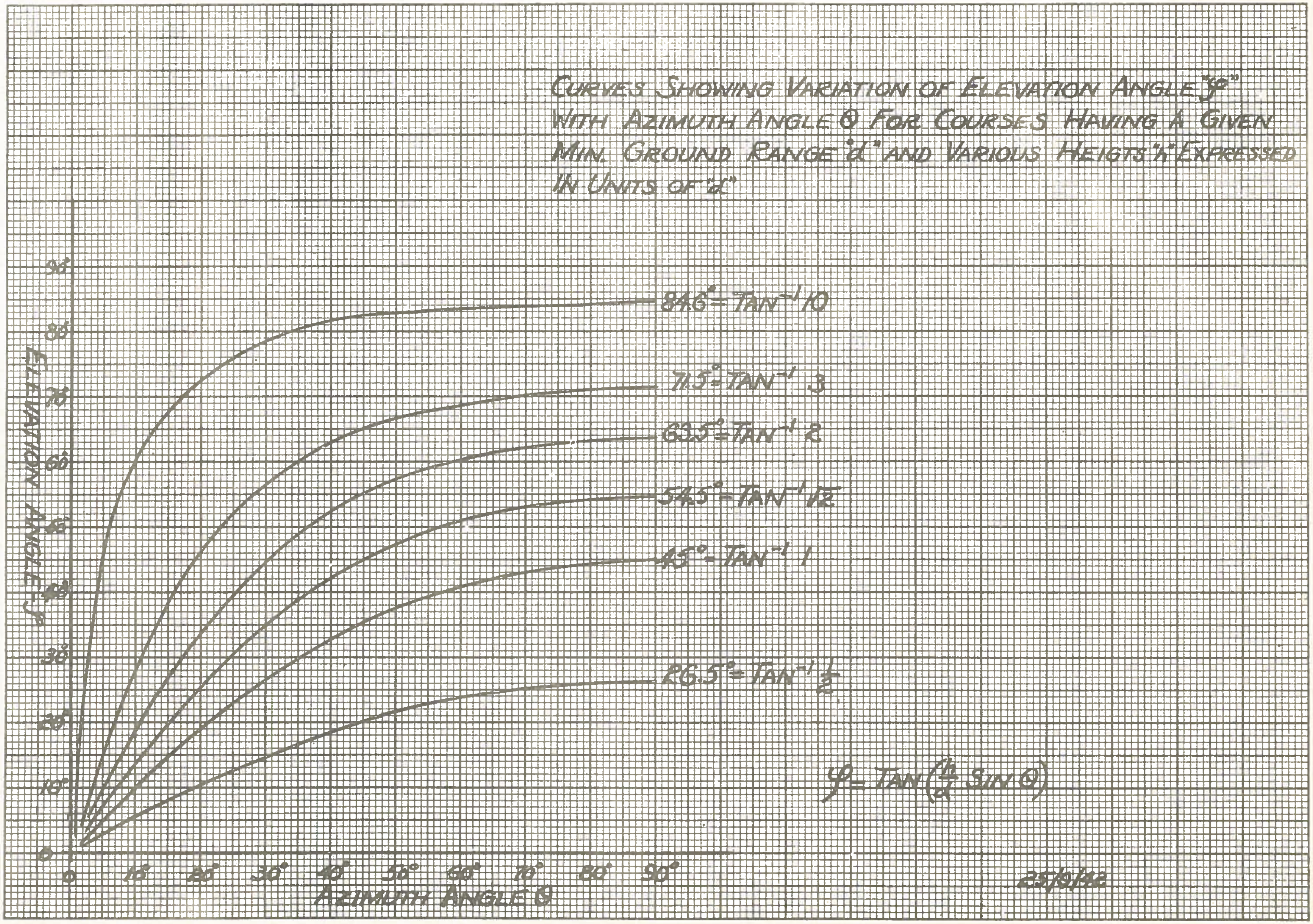
E.L.R. Webb

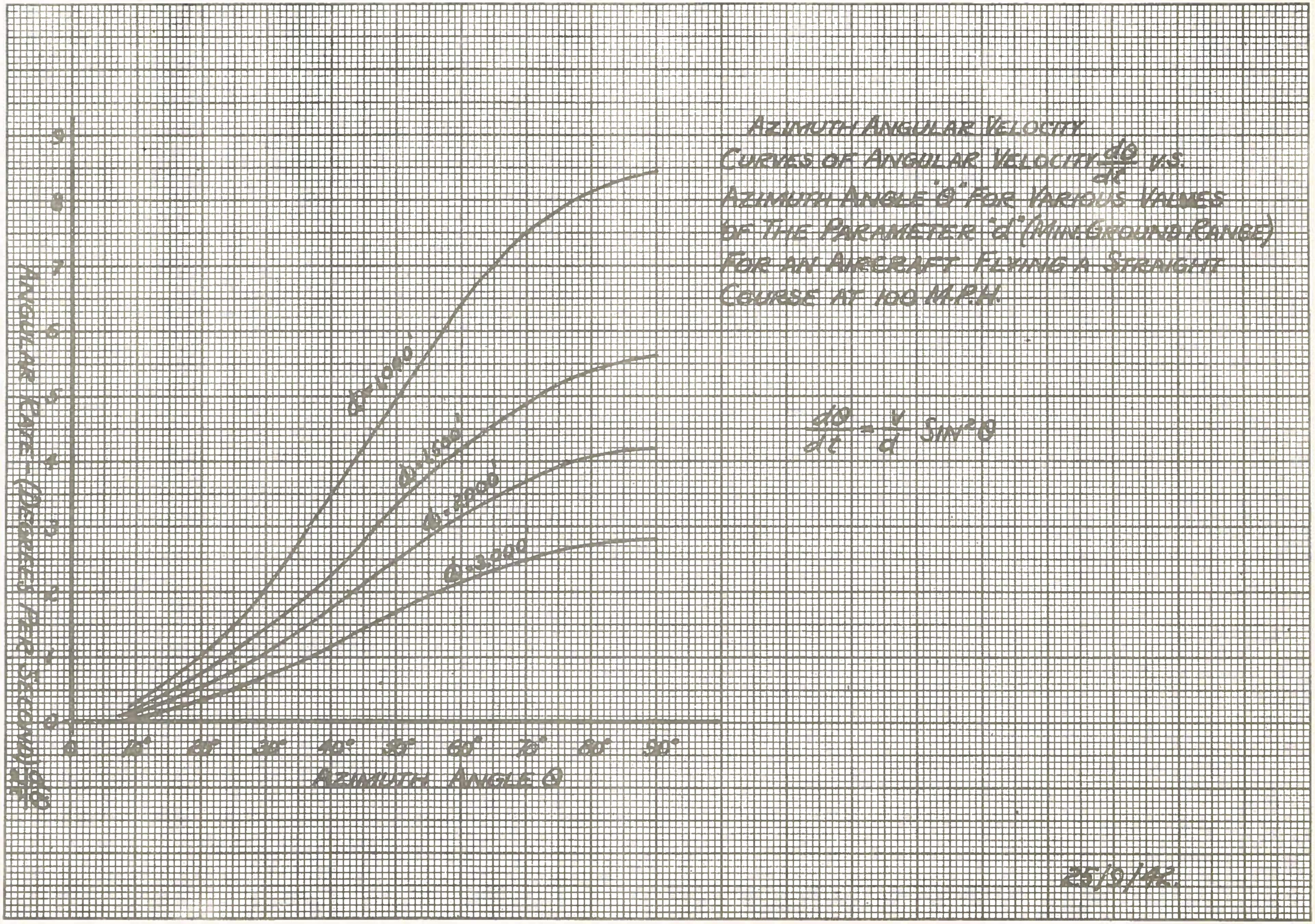
W. McKay
Capt. R.C.A.

OTTAWA
November
1942

REF. No. 330

CURVE





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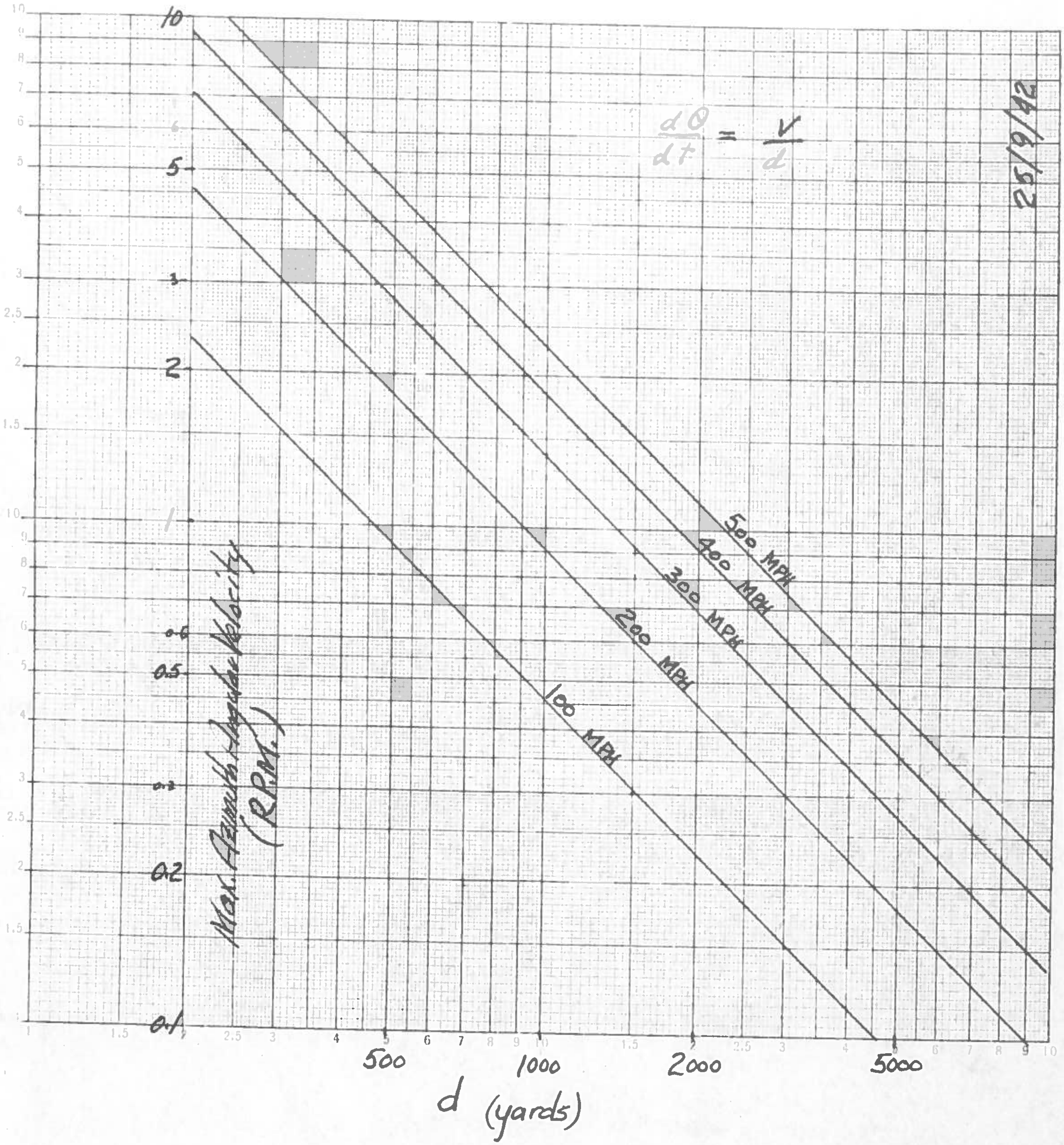

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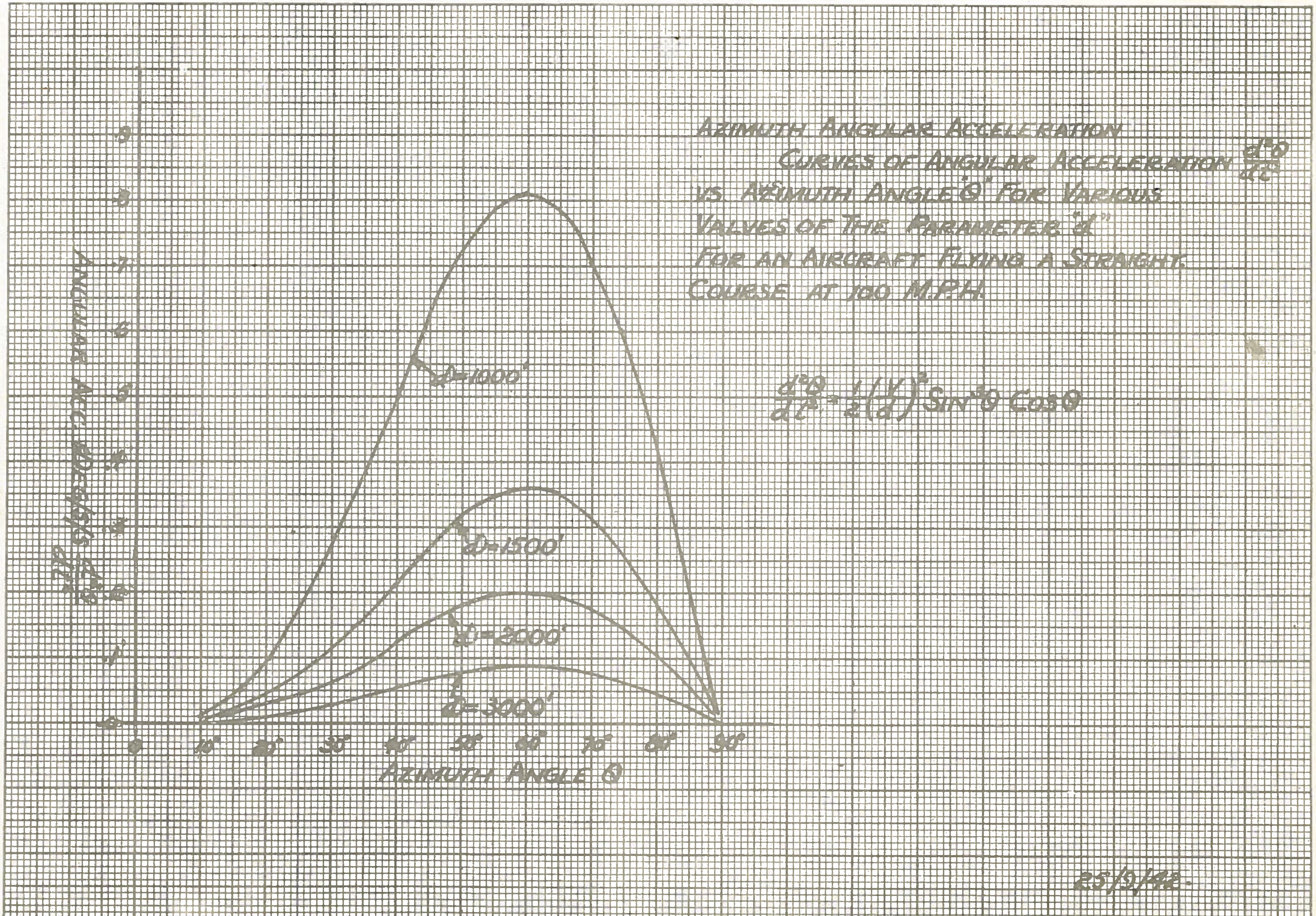

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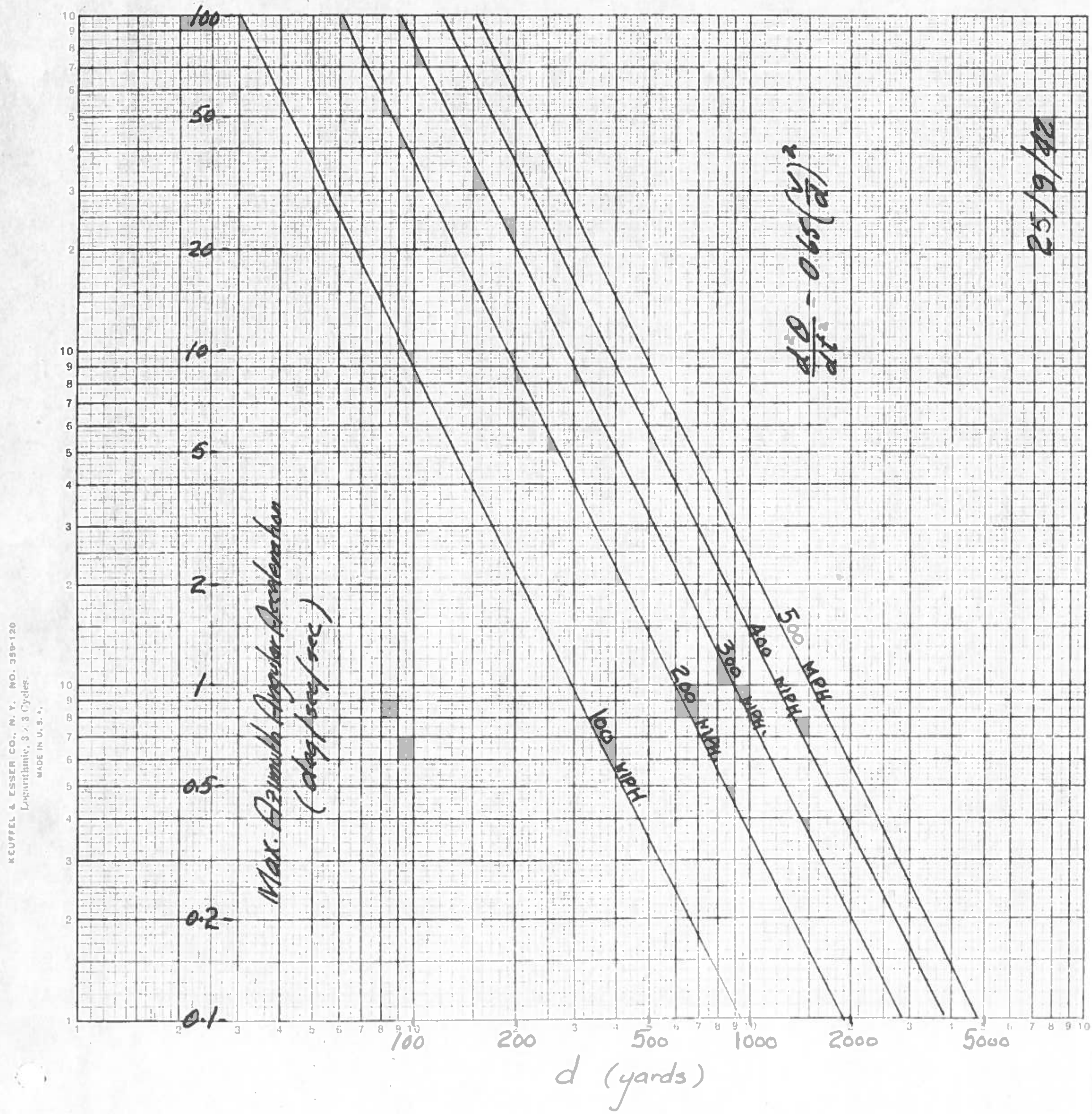

GRAPH 5. Ref. No. 334

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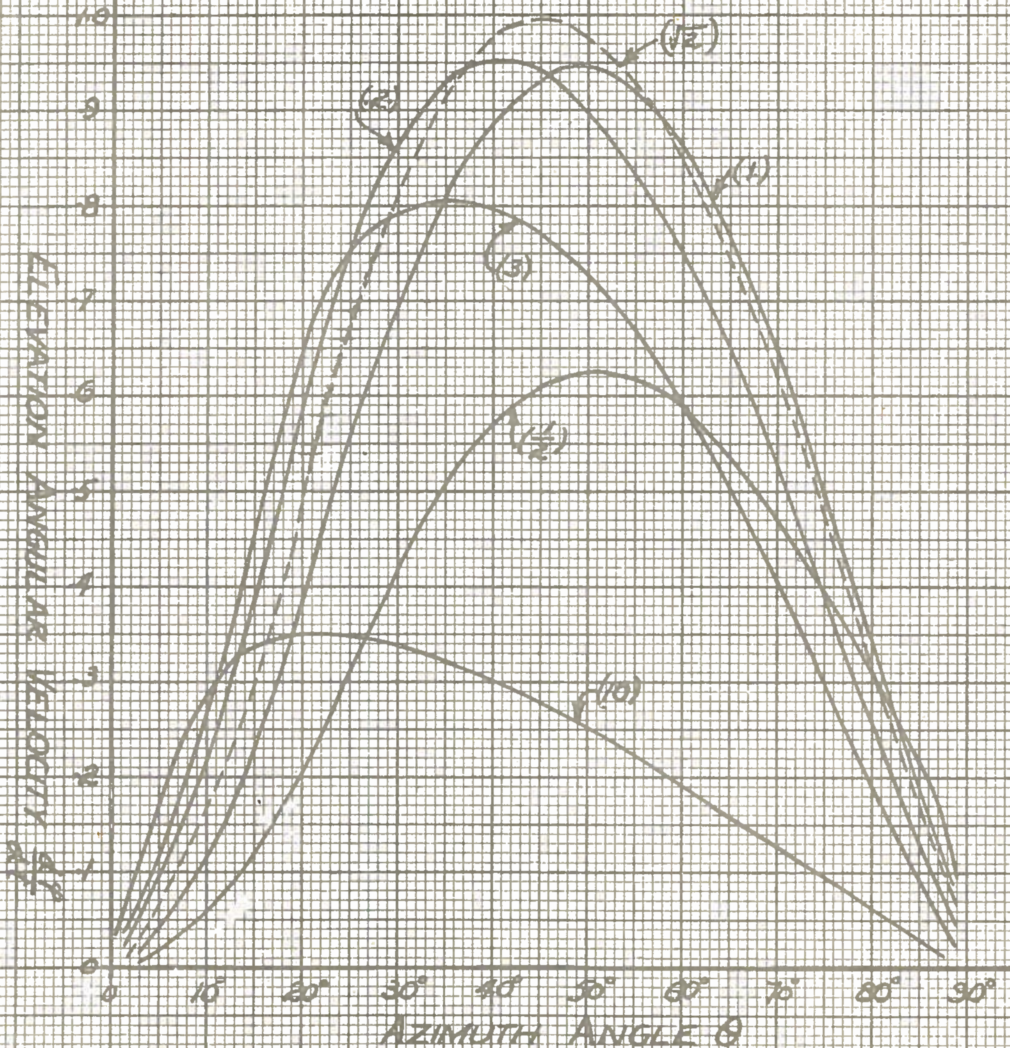


ELEVATION ANGULAR VELOCITY

CURVES SHOWING VARIATION OF ANGULAR VELOCITY "$\frac{d\psi}{dt}$" WITH AZIMUTH ANGLE "θ" FOR COURSES HAVING VARIOUS RATIOS "$\frac{h}{d}$"

$$\frac{d\psi}{dt} = \frac{1}{4}\frac{V}{d} f(\theta)$$

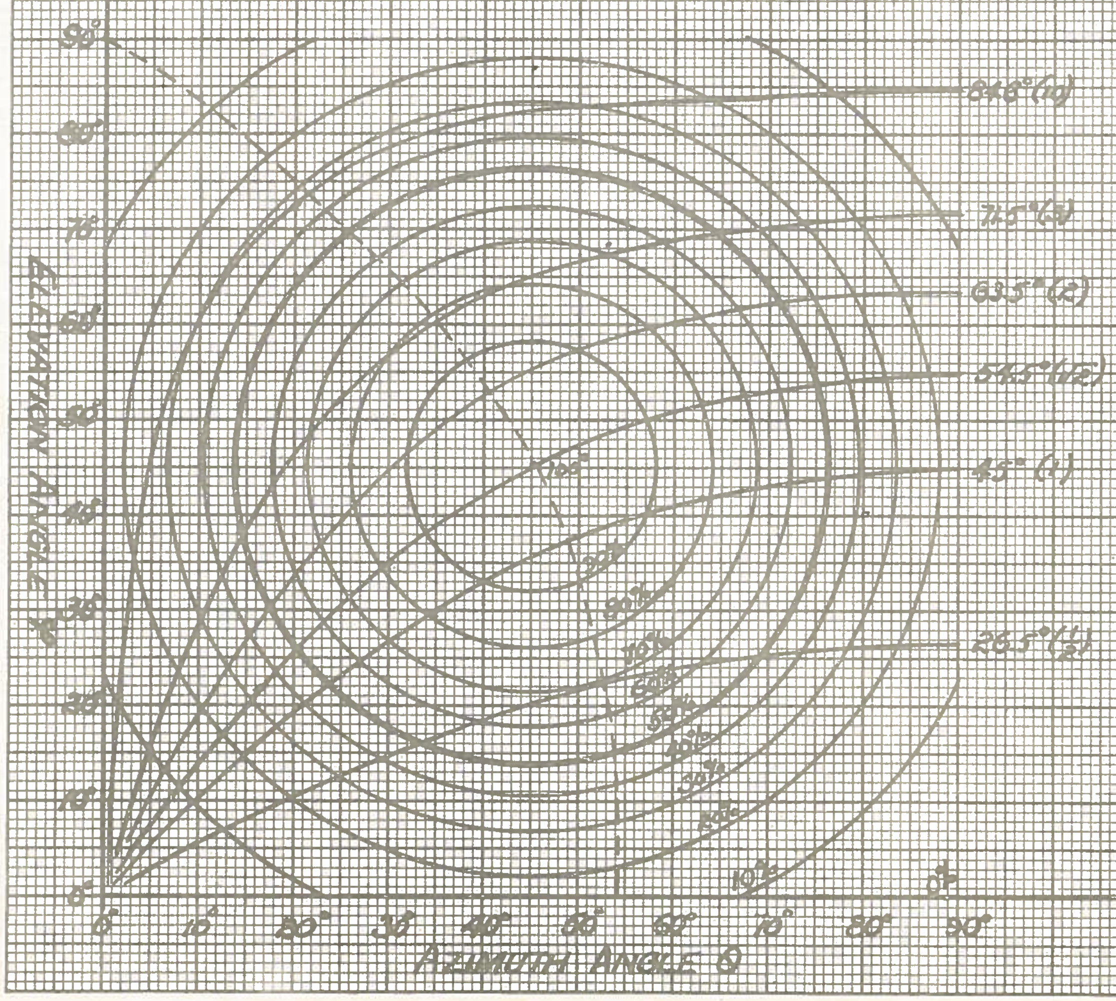
$f(\theta)$ IS PLOTTED HERE

$$f(\theta) = \text{Sin } 2\theta \text{ Sin } 2\left(\text{Tan}^{-1}\left[\frac{h}{d} \text{ Sin } \theta\right]\right)$$

22/9/42.

REF. No. 336

ELEVATION ANGULAR VELOCITY
EQUAL VELOCITY CONTOURS
$100\% = \frac{1}{4}\left(\frac{V}{d}\right)$



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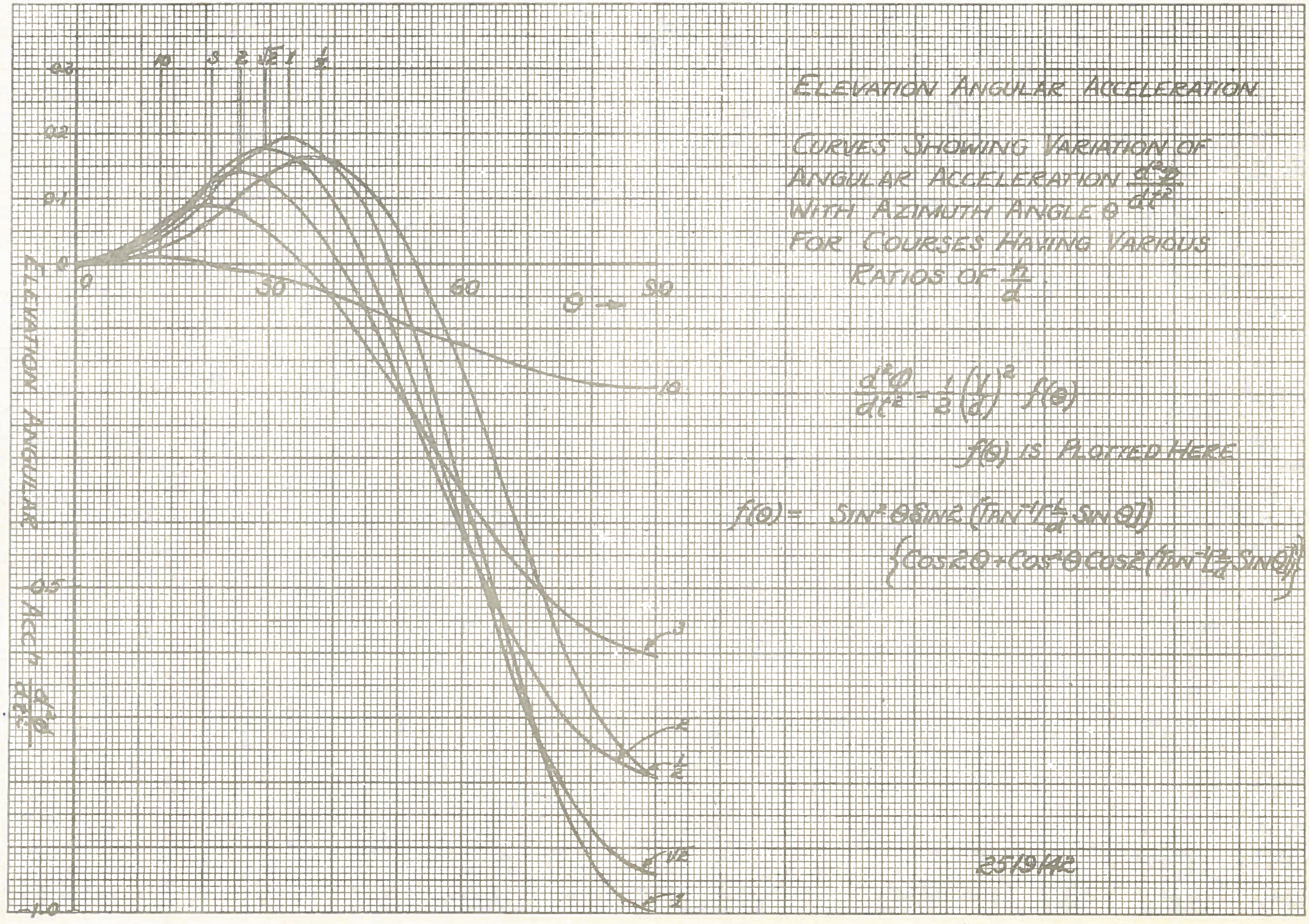
CURVE N

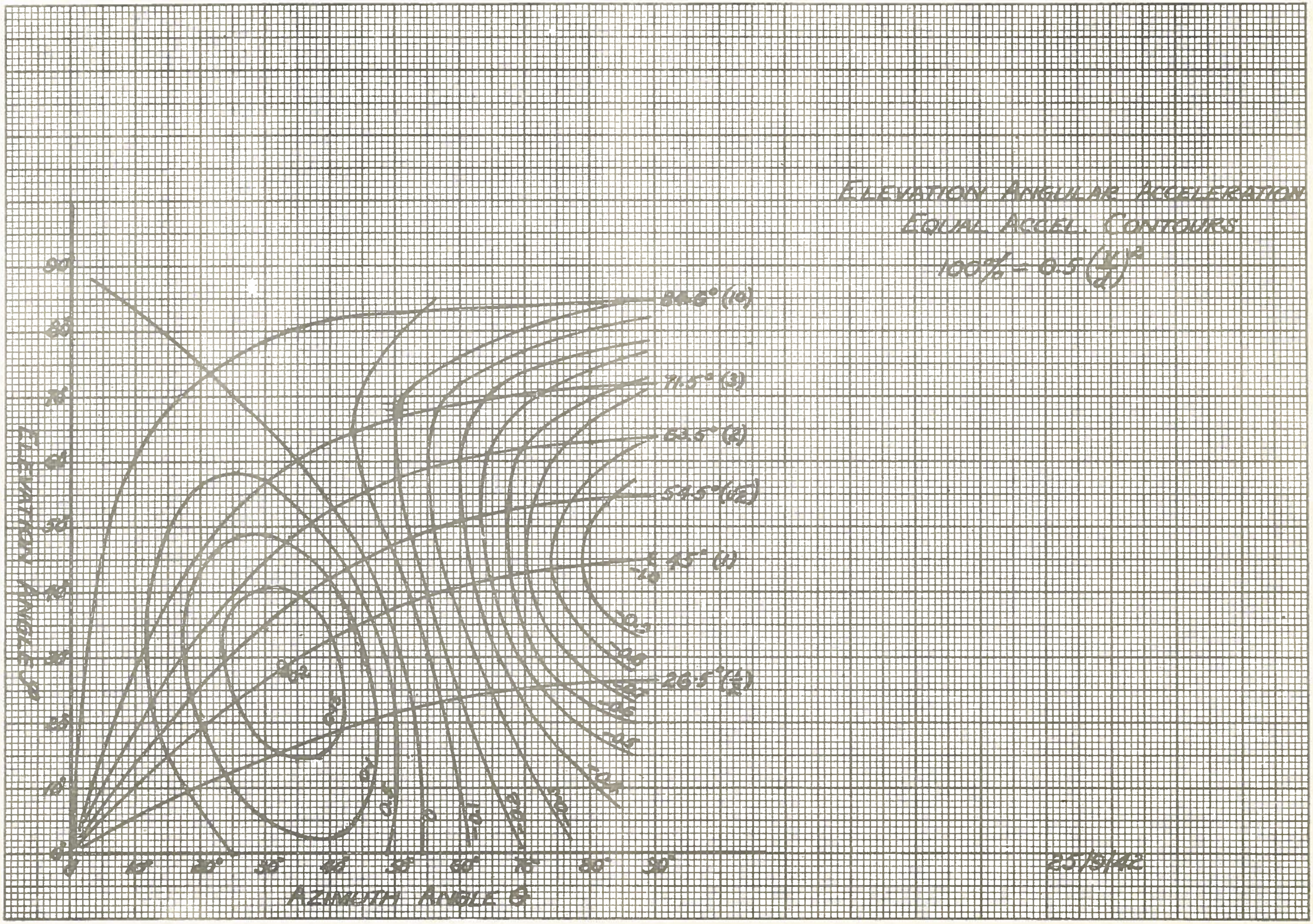

ELEVATION ANGULAR ACCELERATION
CURVES SHOWING VARIATION OF ANGULAR ACCELERATION $\frac{d^2\phi}{dt^2}$ WITH AZIMUTH ANGLE θ FOR COURSES HAVING VARIOUS RATIOS OF $\frac{h}{d}$.
$\frac{d^2\phi}{dt^2} = \frac{1}{2}\left(\frac{V}{d}\right)^2 f(\theta)$
$f(\theta)$ IS PLOTTED HERE
$f(\theta) = \text{SIN}^2\theta\,\text{SIN}2\left(\text{TAN}^{-1}\left[\frac{h}{d}\text{SIN}\theta\right]\right)\left\{\text{COS}2\theta + \text{COS}^2\theta\,\text{COS}2\left(\text{TAN}^{-1}\left[\frac{h}{d}\text{SIN}\theta\right]\right)\right\}$
25/9/42
ELEVATION ANGULAR ACC'N $\frac{d^2\phi}{dt^2}$
0.3
0.2
0.1
0
0.5
-1.0
0
30
60
$\theta \rightarrow$ 90
10
3
2
$\sqrt{2}$
1
$\frac{1}{2}$
10
3
2
$\frac{1}{2}$
$\sqrt{2}$
1

N 7

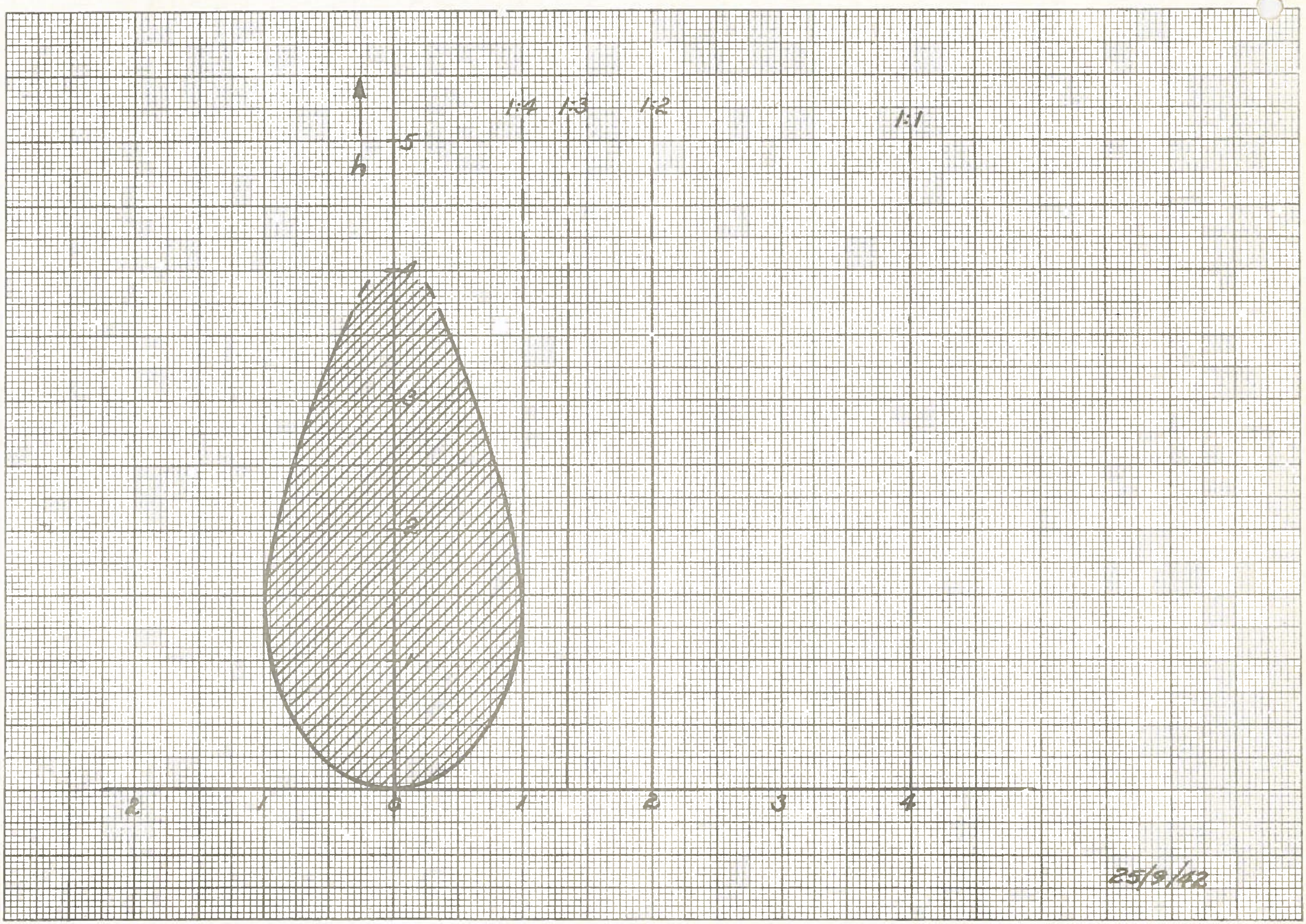
CURVE N

REF. No. 338

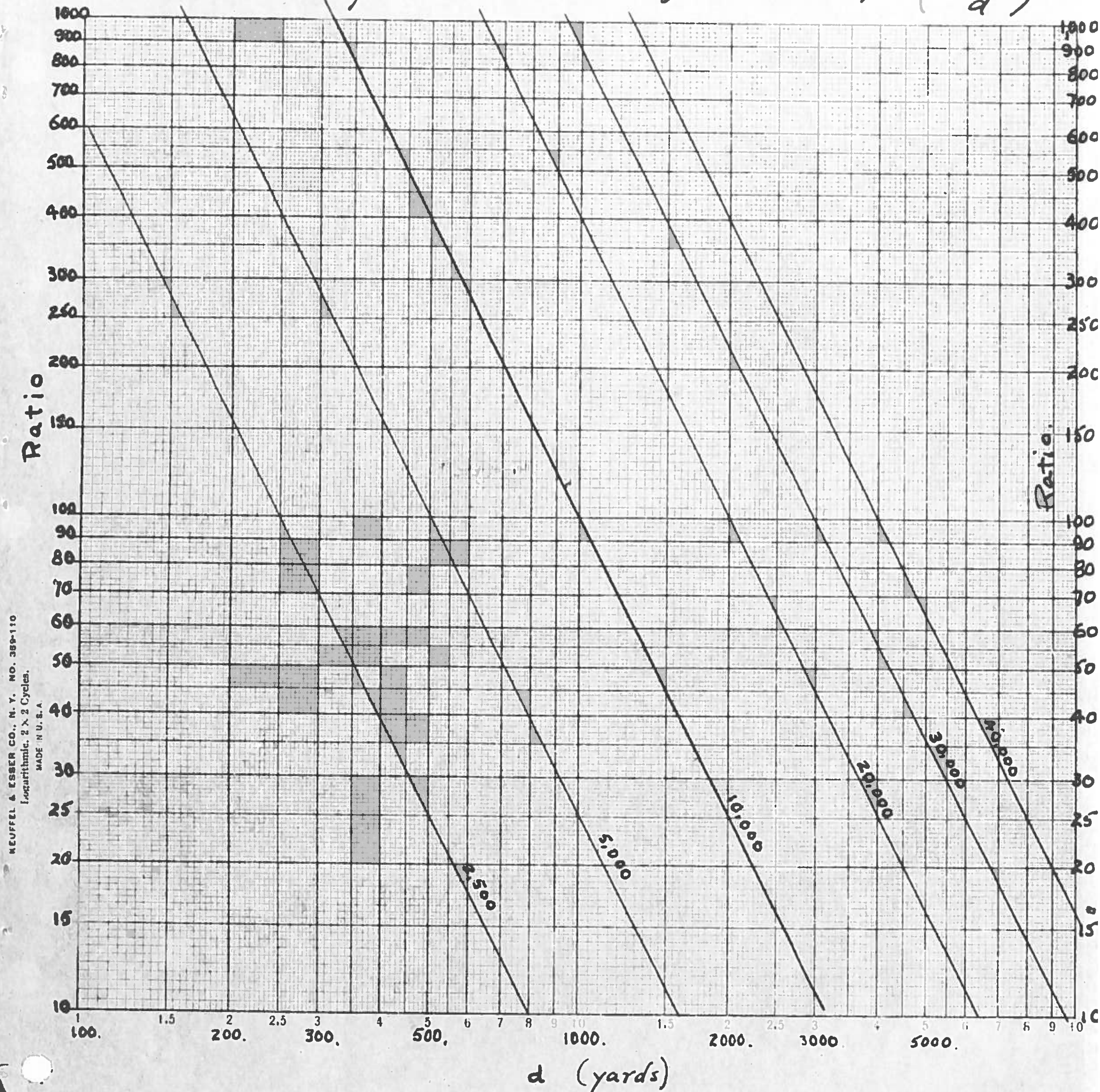
CURVE N



GRAPH 10 REF. NO 339



Ratio of Max. to Min. Azimuth Velocity = $\left(\frac{R_{max}}{d}\right)^2$



GRAPH 11. Ref. No. 340 25/9/42