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INITIAL AND TIME - DEPENDENT DEFLECTIONS OF REINFORCED CONCRETE BEAMS IN THE CRACKED STATE PROPOSALS FOR LIMITS OF DEFLECTION AND SIMPLIFIED CALCULATIONS

BY

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TRANSLATED BY

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PREFACE

The Building Structures Section of the Division has been making a study of reinforced concrete regulations in connection with the work that has been done in preparing the new Reinforced Concrete Design Section for the National Building Code 1960. In the course of this the calculation of deflections in reinforced concrete design has been given attention since the new design standard contains rather more specific regulations than have been given in earlier Canadian documents. At the same time the need for more information on this subject was recognized.

This paper by Dr. Leonhardt of Stuttgart was therefore welcome since it is a valuable contribution in this field. Its existence was brought to the attention of the Revision Committee on Reinforced Concrete of the Associate Committee on the National Building Code which recommended that it be translated and made available for general use by those interested in this field with the specific objective of having it available for future studies of reinforced concrete design regulations.

The Division is grateful to Mr. D.A. Sinclair of the N.R.C. Translations Section for preparing the translation. It is hoped that it will prove of wide use to designers of reinforced concrete structures throughout Canada.

Ottawa, R.F. Legget,

March 1961 Director

NATIONAL RESEARCH COUNCIL OF CANADA

Technical Translation 945

Title: Initial and time-dependent deflections of reinforced concrete beams in the cracked state. Proposals for limits of deflection and simplified calculations

(Anfängliche und nachträgliche Durchbiegungen von Stahlbetonbalken im Zustand II. Vorschläge für Begrenzungen und vereinfachte Nachweise)

Author: F. Leonhardt

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INITIAL AND TIME-DEPENDENT DEFLECTIONS OF REINFORCED CONCRETE BEAMS IN THE CRACKED STATE

PROPOSALS FOR LIMITS OF DEFLECTION AND SIMPLIFIED CALCULATIONS

1. Introduction

The determination of the deflection of reinforced concrete structures in state II*, i.e. when the tension zone is cracked, has hitherto been rather neglected. This is particularly true of the deflection occurring only in the course of time as a result of shrinkage and creep, a deflection that may be two to three times as great as the initial deflection and has therefore led to serious damage on frequent occasions[1]. Serious occurrences have only come about since the raising of the admissible steel and concrete stresses and the application of the slenderness made possible thereby, which is sometimes demanded by architects, but is more often applied by engineers out of a certain pride. The damages which have occurred force us to consider the deformations and limit them in many cases.

There is no lack of theoretical solutions to the calculation of the initial and time-dependent deflections[2-4], but there is lack of pressure to enforce, through an appropriate regulation, deflection calculations and a limitation to slenderness. With such a regulation we must proceed with caution, for on the one hand we do not want to extend unnecessarily the scope of our structural calculations by demanding deflection checks for all beams or slabs, and on the other hand we must not simply restrict the magnitude of the deflection relative to the span, and hence the slenderness, to a certain definite value without taking into account the particular functions of the structural parts in question. For many applications even large time-dependent deflections, e.g. of floor slabs,

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* State I designates the uncracked state. (Trans.)

are of no importance, but if partitioning walls of a brittle material, e.g. plaster board, are erected on a floor slab, then even very slight time-dependent deflections of the slab will result in cracks in the wall and complaints from the owner that we should avoid.

In almost all cases of damage only the time-dependent deflection f_{s+k} is involved. The initial deflection f_o can be taken care of by super-elevation of the forms, and where subsequent loading on finished structures is concerned, by initial loading.

The initial deflection depends primarily on the percentage of reinforcement, i.e. the admissible steel stress (Fig. 1 and 2). The time-dependent deflection f_{s+k}, on the other hand, is determined basically by the factors which influence the shrinkage (s) and creep (k) of the concrete, i.e. the water and paste content of the concrete, the relative atmospheric humidity in its vicinity and the value of the concrete stress σ_b* under the continuous load q_D. We shall use the usual coefficients, namely the total shrinkage allowance ε_s and the total creep number φ, which are chosen for the different kinds of concrete and different environmental conditions in a manner similar to that applied in the case of prestressed concrete[5]. However, the time-dependent deflection f_{s+k} is almost entirely independent of the steel stress σ_e.

2. Suggestion for a Deflection Regulation

If the time-dependent deflections are generally a decisive factor in causing damage, then it seems reasonable to employ them as a criterion for restricting the slenderness or for the requirement of a deflection check. This involves primarily an estimation of the amount of time-dependent deflection which can be tolerated. If we assume for this $f_{s+k} = \frac{1}{500} l$, i.e. in the case of a slab with

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* In this paper coefficients with the subscript b refer to concrete (Beton); similarly e refers to steel (Eisen). (Trans.)

a 5 m span a time-dependent deflection of 1 cm (already a considerable amount), we obtain the slenderness values l/h shown in Table I for structures in the open (ε_s = 0.30 mm/m, φ = 2) and in heated buildings (ε_s = 0.45 mm/m, φ = 4) for two concrete quality classes B160/225 and B300/450, which have been combined for the sake of simplicity.

These slenderness values are everywhere less than the previous limits of 35 for one-way slabs or 50 for two-way slabs. That is to say, at the now permissible slenderness we must expect, even for low concrete stresses, time-dependent deflections greater than $l/500$. The f_{s+k} values rise steeply with the concrete stress σ_D under a continuous load q_D, so that, for example, at σ_b = 80 kg/cm^2 for structural members in buildings, slendernesses of 11 and 13 already yield the cited limit of $l/500$. Generally speaking the continuous load q_D is only part of the design load q, so that the concrete stresses due to q_D do not attain too high values, but remain considerably below the permissible σ_b, which, moreover, is often not utilized. For the most part, therefore, the small slenderness values given for high concrete stresses are not, in practice, critical.

It will be conceded that $l/500$ as a time-dependent deflection is already a generous allowance, up to which we ought to be spared a mathematical check of the time-dependent deflection. However, if larger time-dependent deflections are expected, then in future these should probably be calculated wherever they are not clearly harmless, so that the engineer will be aware of them and will discuss the conclusions to be drawn from them with the owner or his representative. In the case of buildings the architect should share the responsibility for this decision, because often the engineer does not know what installations may be planned that will be sensitive to deformations.

An absolute slenderness limit is necessary, as before. This should also be made to depend on the time-dependent deflection and should not be petty, since in many applications even large time-dependent deflections are harmless. As a limit for the time-dependent deflection we might choose $\ell/250$ or $\ell/200$, since, for example, for roof slabs of 2 m span a time-dependent deflection of 1 cm would be just barely acceptable if the slope were chosen great enough. It is essential, however, that in future the engineer should be made aware of the magnitude of this time-dependent deflection by the required deflection determination and should take it into account in the design stage.

If this comes about then the previous absolute slenderness limits can be retained, i.e. $h = \ell/35$ or $\ell/40$ for slabs or beams stressed in one direction and $\ell/50$ for slabs or ribbed slabs stressed in two directions. Here ℓ refers to simple beams with freely rotatable supports; other conditions of support may be taken into account approximately by introducing the slenderness ℓ_i/h in place of ℓ/h with

$\ell_i = 1.0\ \ell$ for the single span simply supported beam

$\ell_i = 0.8\ \ell$ for end spans
$\ell_i = 0.7\ \ell$ for central spans } for continuous beams with equal spans

$\ell_i = 0.6\ \ell$ for full fixity on both ends

$\ell_i = 1.7\ \ell_k$ for cantilevered sections fully fixed (with ℓ_k = cantilever length)

$\ell_i \leq 2.2\ \ell_k$ for cantilevered beams partially fixed (distributed load)

For two-way slabs practically the same ℓ_i values hold as for beams and one-way slabs. The smaller deflection of these members is due to the distribution of the moments into M_x and M_y.

The values ℓ_i can be calculated in a simple manner. The resultant length ℓ_i of a beam for any given degree of fixity and any given load is the length of the simple beam simply supported at both ends and uniformly loaded, the deflection of which is equal to the deflection of the loaded beam and the maximum moment of which is equal to the maximum positive moment of this beam (Fig. 3).

We may take $f = \frac{k_d M l^2}{EI} = \frac{5 M l_1^2}{48 EI}$, for k_d see Section 3.

Hence, with 48/5 = 9.6 we obtain

$$\boxed{l_i = l\sqrt{9{,}6\, k_d}}$$

In determining l_1 West's(13) method can be advantageously used for the determination of the deflections.

According to West the following is obtained for the continuous beam:

$$f = \frac{5\, l_n^2}{48\, EI}\left(M_m + \frac{M_i + M_k}{10}\right).$$

(The moments must be given their correct sign.)

Hence

$$\boxed{l_i = l_n \sqrt{1 + \frac{M_i + M_k}{10\, M_m}}}$$

For cantilevered beams we obtain

$$f = \frac{l_k}{12\, EI}\left[-4\, M_m\, l_n - M_k\,(2\, l_n + 3\, l_k)\right].$$

Hence

$$\boxed{l_i = l_k \sqrt{\frac{48}{60}\,\frac{4\, M_m \alpha + M_k\,(2\alpha + 3)}{M_k}}}$$

For uniformly distributed load and a single-span beam with cantilevered arm the following values are obtained for l_1/l_k as a function of $\alpha = l_n/l_k$ (Fig. 6).

For concentrated loads on the end of the cantilever the values shown in Fig. 7 are obtained.

Fig. 8 and 9 show the slendernesses l/h at which the limiting values for the time-dependent deflection are reached. The lower curves (1) refer to the limit $f_{s+k} = l/500$, the upper curves (2) to the limit $f_{s+k} = l/200$. It will be noted that the admissible steel stresses have only moderate effect for higher concrete strengths B300/450 and none at all at B160/225. Thus, as far as the time-dependent deflections are concerned high admissible steel stresses are not harmful. The concrete stress σ_b and the creep coefficients φ, on the other hand, are of great importance.

With the chosen limits, we get the slenderness limits in Table II; here σ_b is the concrete stress for n = 15 due to q, q_D = continuous load, q = design load.

The a-columns give q_D/q times slendernesses l_1/h for the chosen concrete strength depending on the design stress σ_b (calculated with n = 15) due to q, for which the time-dependent deflection due to q_D reaches the value $l_1/500$. The b-columns give the absolute limiting value of q_D/q times slendernesses (i.e. corresponding to $l_1/200$).

When the ultimate load design method is used the following value

$$\frac{M}{b\,h^2} k_\mu \text{ with } k_\mu = \frac{2}{k_x\left(1 - \frac{k_x}{3}\right)} = \frac{6}{k_x(3 - k_x)},$$

which depends only on cross-sectional values and E_e or E_b, replaces σ_b;

since

$$k_x = \mu \frac{E_e}{E_b}\left\{-1 + \sqrt{1 + \frac{2}{\mu \frac{E_e}{E_b}}}\right\} \text{ and } \mu = \frac{F_e}{b\,h}.$$

The initial deflections in the cracked condition should also be calculated for conditions between columns a and b and members with l > 4 m should be cambered to provide for these deflection values and, if necessary, the camber should be increased to allow for the time-dependent deflections.

Finally, the value to be taken for q_D should be determined. If the continuously acting live load is not known, then a minimum of $q_D = g + 0.2p$* should be assumed, since a small part of the live load will probably be continuously active in almost every case.

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* g = uniformly distributed dead load.

p = uniformly distributed live load.

3. Simplified Calculation of Initial and Time-Dependent Deflections

3.1 Initial deflection

Let us consider the strains in an element of a beam of rectangular cross-section and without compressive reinforcement in the cracked state. Assuming that the cross-sections remain plane, then according to Fig. 10 the following relationships apply: Radius of curvature of the elastic curve $\frac{1}{\rho} = \frac{\varepsilon_e}{h - x} = \frac{\varepsilon_b}{x}$.

On the basis of the steel strain, we may also write

$$\varepsilon_e = \frac{\sigma_e}{E_e} = \frac{M}{E_e F_e z};$$

and hence

$$\frac{1}{\varrho} = \frac{M}{E_e F_e z (h - x)}.$$

By analogy with the known equation for homogeneous building materials $\frac{1}{\rho} = \frac{M}{EJ}$, we conclude that the denominator

$$\boxed{E J^{(II)} = E_e F_e z (h - x)} \qquad (1)$$

can be taken as the flexural stiffness of steel reinforced concrete beams in the cracked state and can be used in the treatment of statically indeterminate structures.

K. Jäger(6) has investigated this in detail and found that the values for the flexural stiffness which apply to the region of positive or negative maximum moments with their corresponding reinforcements are applicable in every case without alteration over the entire moment zone.

If $\frac{1}{\rho}$ is known, the elastic curve of a beam can be determined in the well-known manner according to Mohr as the moment diagram of the beam loaded with the area $\frac{1}{\rho}$ for the appropriate moment distribution, and variations in reinforcement can be taken into account.

For the beam resting on two supports, with normal reinforcement (bent-up rods) we find from the example of Fig. 11 that with decreasing reinforcement and the corresponding curve of x, and hence also $z = h - \frac{x}{3}$, the curvature $\frac{1}{\rho}$ on the entire length of the beam is by no means affine to the moment diagram but decreases very little, so that as the elastic curve we get almost a parabola with the maximum ordinate

$$f_0 = \frac{1}{8} \frac{Ml^2}{E_e F_e z (h - x)},$$

where M, F_e, z and x are valid for the section at mid-span.

When this value is compared with experimental ones it is found to be too large, probably because the uncracked state is maintained in the outer ends of the beam and because the concrete between the cracks, at least during application of the first tensile stresses, contributes some tensile strength. Better agreement with the experimental values is obtained, therefore, by neglecting the decrease of reinforcement due to bent-up rods and thus using the deflection coefficients k_d derived from the shape of the moment diagram, which can be found in any handbook. The contribution of the concrete can be taken into account by means of a reduction coefficient α ranging from α_o = 0.75 to 0.9, depending on the strength of the concrete and the bonding characteristics of the reinforcement(7). The initial deflection is thus generally obtained as

$$\boxed{f_0 = \alpha_0 k_d \frac{M l^2}{E_e F_e z (h - x)}} \qquad (2)$$

where α_o - reduction coefficient for taking into account the contribution of the concrete to the tensile strength, (tentatively for plain rods, 0.9; for deformed rods, 0.75)

k_d - deflection coefficient depending on the moment diagram from $f = \frac{k_d M l^2}{EJ}$; for uniformly loaded simply supported beams, $k_d = \frac{5}{48}$

M - moment due to initial load at the time t = 0

F_e - reinforcement for maximum M due to q

z and x are valid for the initial modulus of elasticity of the concrete for short term loading E_{bo} and at the point of maximum M.

When we now compare an experimental curve of loading deflection with the curve according to the above equation (Fig. 12) we find a departure in the initially steep part of the curve corresponding to the uncracked state. Rabich[3] dealt with this zone separately. However, at high permissible steel stresses the zone is generally small, and furthermore the initial reduction of the deflections is generally lost after a few changes of load, so that one is probably justified in considering the curve of deflection for live loads as a straight line originating at zero. The curvature of this line under higher concrete stresses due to the curvature of the E_b curve, which Jäger[2] reported as close to reality, is of no interest for practical purposes.

Developing the equation $\frac{1}{\rho} = \frac{\varepsilon_b}{x}$ from the concrete shortening, then

$$\varepsilon_b = \frac{\sigma_b}{E_b}, \quad \sigma_b = \frac{2\,D}{b\,x}, \quad D = \frac{M}{z},$$
$$\frac{1}{\varrho} = \frac{2\,M}{E_b b x^2 z}.$$

These theorems assume straight line increase of the stresses σ_b or a constant E_b, which holds with sufficient accuracy for live load stresses $\sigma_b < \frac{\beta_p}{2.5}$.

We thus obtain a second expression for the flexural stiffness of reinforced concrete beams

$$\boxed{E\,J^{(\mathrm{II})} = E_b\,b\,x^2\,z} \tag{3}$$

and accordingly, for the initial deflection

$$\boxed{f_0 = \alpha_0\,k_d\,\frac{2\max\,M\,l^2}{E_{b0}\,b\,x^2\,z}} \tag{4}$$

3.2 Deflection due to shrinkage

In determining the deflection due to shrinkage it is generally assumed that the outer compressive fibre of the beam is shortened by the entire amount of the shrinkage ε_s, while in the zone of tension the shrinkage takes place between the cracks without appreciable effect on the steel strain. From Fig. 13, therefore, we obtain the radius of curvature of the elastic curve for shrinkage alone

$$\frac{1}{\rho_s} = \frac{\varepsilon_s}{h}.$$

This would apply over the entire length of the beam if the outer zones of the beam did not remain in the uncracked state, so that there a considerably smaller curvature occurs, which depends on the prevention of shortening due to shrinkage by the reinforcement. The zones remaining in the uncracked state depend on the moment diagram, so that the deflection due to the shrinkage can be determined approximately with the same coefficients k_d as are used for the deflection due to loading. Hence

$$\boxed{f_s = k_d \frac{\varepsilon_s l^2}{h}} \qquad (5)$$

Here ε_s has its actual value of 0.30 to 0.50 mm/m instead of the nominal value of only 0.15 mm/m found in many codes. ε_s can be decreased only if the beam is kept moist for months[5]; however this practically never occurs.

3.3 Deflection due to creep

Only a few test results are available on deflections due to concrete creep[8-11]. The conditions in the cracked state are not so simple as in the uncracked state, for example in the case of prestressed concrete, where for the creep deformation we simply put $\varepsilon_k = \varphi \cdot \varepsilon_o$, because in the cracked state the creep affects the position of the neutral axis. The latter is displaced downward, i.e. the compression zone becomes greater as the concrete stresses decrease, but the steel stresses become greater because the inside lever arm becomes smaller. As an example of this we may cite the shifting of the neutral axis in deformation diagrams in the course of an 18-month period of loading in the open air according to the test results of Hajnal-Konyi[9] (Fig. 14).

The creeping of the concrete is approximately proportional to the stress and is therefore equivalent to a reduction of the E_b modulus. The effect of the creep may therefore be expressed by

$$E_{b\infty} = \frac{E_{b0}}{(1 + \varphi)},$$

when the initial deformation is taken into account.

If we use the low $E_{b\infty}$ modulus, then the lowest position of the null line and the higher steel stress are obtained. The entire initial deflection plus that due to creep is now obtained with $E_{b\infty}$ from equation (4), provided z and x with $E_{b\infty}$ are also determined. At the same time M_D due to q_D, i.e. due to the permanent load producing creep, must also be used.

$$\boxed{f_\infty = \alpha_\infty\, k_d \frac{2\, M_D\, l^2}{E_{b\infty}\, b\, x_\infty^2\, z_\infty}} \qquad (6)$$

α_∞ is a correction factor taking into account the tension contribution of the concrete after load cycles or continuous loading, depending on the quality of the bond. No test results are as yet available for this value and it can therefore only be estimated, assuming $\alpha_\infty = 1.0$ for the time being, for the case of plain round rods, and $\alpha_\infty = 0.9$ for deformed steel.

3.4 Simplified determination of deflections

For a simplified determination of the deflection components f_o and f_{s+k} it is necessary to use terms which occur directly in the static calculation, i.e. the determination of x is to be avoided. This is all the more necessary when it is considered that the k_x* values of the standard design procedure according to DIN 4224 hold only for n = 15, i.e. E_b = 140,000 kg/cm², i.e. for a value which does not correspond to the actual E_b values of present-day concrete qualities. The simplified calculation must also be applicable when the ultimate load method is shortly introduced.

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* Here k_x is to be calculated for the actual n and not n = 15, therefore is not equal to k_x according to DIN 4224.

These conditions are satisfied by starting from the cross-sectional values b, h, F_e** or μ** and from the actual moduli E_e and E_b. At the same time let us assume as an approximation that the $\sigma - \varepsilon$ curve of the concrete is a straight line for low live-load stresses.

Since the position of the neutral axis depends only on μ, E_e and E_b, all the relationships with E_e = 2100 tons/cm^2 as constant can be taken together in a table of curves by rewriting the denominator of equation (2) as follows:

$$E_e F_e z (h - x) = E_e F_e h^2 \left(1 - \frac{4 k_x}{3} + \frac{k_x^2}{3}\right) = F_e h^2 k_e$$

$$\boxed{k_e = E_e \left(1 - \frac{4 k_x}{3} + \frac{k_x^2}{3}\right)} \qquad (7)$$

k_e is plotted in Fig. 15 as a function of μ for various values of E_b.

We then obtain from (2) the initial deflection due to a load which results in the maximum moment M

$$\boxed{f_0 = \frac{\alpha_0 k_d M l^2}{\mu b h^3 k_{e0}}} \qquad (8)$$

if k_{eo} is read for E_{bo}.

The total deflection after conclusion of shrinkage and creep is

$$\boxed{f_\infty = \frac{\alpha_\infty \cdot k_d l^2}{h} \left(\frac{M}{\mu b h^2 k_{e\infty}} + \varepsilon_s\right)} \qquad (9)$$

if $k_{e\infty}$ is read for $E_{b\infty} = \frac{E_{bo}}{1 + \varphi}$.

The time-dependent deflection is

$$\boxed{f_{s+k} = f_\infty - f_0} \qquad (10)$$

Here the equivalent moment is to be substituted for f_o and f_∞.

The flexural rigidity is

$$\boxed{E J^{(II)} = \mu b h^3 k_e = F_e h^2 k_e} \qquad (11)$$

Similar formulae can be developed from equation (4) with a coefficient k_b. However, the curves for k_b are less suitable for reading than those for k_e.

** F_e = cross-sectional area of steel.

μ = percentage of steel reinforcement to concrete cross-section. (Trans.)

We thus have a very simple method of determining the deflections from the cross-sectional values without proceeding via the stresses, the loads g or q_D being taken into account by substitution of the relevant M_g or M_D values, the climatic conditions by ε_s and φ, the concrete qualities by the corresponding E_{bo} value and the final curing of the concrete, or its state at removal of the forms by correction factors k_s for ε_s and k_1, k_2 for φ according to reference (5). It must be realized, however, that the results are only approximate values, since the properties of the concrete vary considerably and relevant test results are still scarce.

3.5 Comparison of the simplified calculation with test results

Tests by Sattler[8] with rectangular beams of B300, E_{bo} = 280,000 kg/cm^2, duration of loading about 80 days in the laboratory, concrete steel IIIb

Test	l m	b/d cm	μ %	$g + p$ t/m	l/h	φ	ε_s	f_0 cm Calc.	f_0 cm Meas.	f_∞ cm Calc.	f_∞ cm Meas.
a	4,0	10/16	0,72	0,137	30	2,3	$30 \cdot 10^{-5}$	1,7	1,6	3,0	3,2
b	2,1	100/7,5	0,64	0,504	38	1,8	$20 \cdot 10^{-5}$	1,2	0,9*	1,9	1,8

* Estimated

Tests in Smethwick by Hajnal-Konyi on rectangular beams b x d = 12.7 x 19 cm, duration of load about 560 days in the open

Series	Concrete	l m	$\frac{l}{h}$	σ_e t/cm²	μ %	ε_s °) 10^{-5}	φ °)	f_∞ cm Calc.	f_∞ cm Meas.	Result α_∞
1	B 225 E_{bo} = 200 000	6,40 4,81 3,20	40 30 20	1,4	1,2	20	2,0	5,0 2.8 1,2	5,2 2,8 1,1	~1,0
	B 300 E_{bo} = 280 000	6,40 4,81 3,20	40 30 20	Plain bars				4,3 2,4 1,1	4.8 2,4 1,0	
2	B 225	6,40 4,81 3,20	40 30 20	2,4	0,65	20	2,0	6,6 3,7 1,6	5,6* 3,1* 1,3*	 0,84 0,81
	B 300	6,40 4,81 3,20	40 30 20	Tentor deformed bars				5,9 3,3 1,5	5,9 2,7 1,2	1,0 0,87 0,80

° Not measured

* Concrete strength values presumably better than B225, cf. test results with B300

The series of tests are continuous tests without load cycling, which explains why α_∞ is low for the case of deformed bars. With repeated load greater deflection would occur, therefore also greater values of α_∞, closer to 0.9. Unfortunately in these tests ε_s and φ were not measured, but the values included correspond to test results for similar conditions and times.

4. Deflections of T-beams

The above equations were derived for rectangular cross-sections. In the case of T-beams fundamentally the same considerations apply, and therefore equations (1) and (2) again hold.

From equation (2), after substitution of μ, k_z and k_x we find

$$f = \alpha \cdot k_d \cdot \frac{M\, l^2}{b \cdot h^3\, \mu\, k_e} \text{ with } k_e = E_e \cdot k_z\,(1 - k_x)\,.$$

The coefficient k_e must be determined anew for k_x and k_z of the T-beam. These can be written with the following factors

$$b_0 = \alpha_b \cdot b\,, \qquad F_e = \mu \cdot b \cdot h\,, \qquad \alpha_d = \frac{d}{x}\,,$$

$$d = \delta \cdot h\,, \qquad n = \frac{E_e}{E_b}\,, \qquad x - d = \alpha_x \cdot x\,,$$

$$k_x = \frac{n \cdot \mu + \delta\,(1 - \alpha_b)}{\alpha_b}\left[-1 + \sqrt{1 + \frac{[\delta^2\,(1 - \alpha_b) + 2\,n\,\mu]\,\alpha_b}{[n\,\mu + \delta\,(1 - \alpha_b)]^2}}\right],$$

$$k_z = 1 - k_x/3 \text{ for } k_x < \delta\,.$$

$$k_z = 1 - \frac{\delta}{\alpha_d}\left(1 - \frac{2}{3} \cdot \frac{1 - (1 - \alpha_b)\,\alpha_x^3}{1 - (1 - \alpha_b)\,\alpha_x^2}\right) \text{ for } k_x > \delta\,.$$

Fig. 16 shows a comparison of the case of k_e values for T-beams and rectangular cross-sections.

The deflections of the slab beam cross-sections are greater with increasing μ than for the rectangular cross-section, since k_e is in the denominator. For the case of low E_b value, i.e. in the event of large creep leading to a lower $E_{b\infty}$ value, the deflections may be up to twice as great.

The comparison shown refers to a definite T-beam cross-section with

$$b/b_o = 5,$$

$$h/d = 8.$$

Fig. 17 shows the change in k_e for a fixed value E_b and μ, if (a) h/d and (b) b/b_o are introduced as variables. These changes are only slight for $\mu = 0.5\%$. They increase with increasing μ in the same ratio as the differences shown in Fig. 16 between the k_e values of the rectangular cross-section and the T-beam.

The time-dependent deflections of slab beam cross-sections are determined exactly as for the rectangular cross-section. Here only the k_e values of the T-beam cross-section in question are used.

These results, obtained from purely theoretical considerations, should be tested experimentally.

5. Effect of Compressive Reinforcement on the Deflection of Rectangular Cross-sections

Tests by Washa and Fluck[11] show that beams with compressive reinforcement are subject to considerably smaller time-dependent deflections than beams without compressive reinforcement. The initial deflections, on the other hand, are almost the same.

The influence of compressive reinforcement can be taken into account approximately by conversion of the differential equations set up by Dischinger[12] to determine the effect of longitudinal reinforcement on the creep deformation of columns. In this the reduced creep coefficients φ' is determined and is then substituted for φ in the deflection calculation.

If a straight line distribution of stress be assumed and the compressive reinforcement is referred to the cross-section $x' \cdot b$, where $x' = \frac{x_o + x_\infty}{2}$ (Fig. 18), and if it be assumed also that the deformation due to shrinkage follows a course similar to the creep deformation, then the reduced creep coefficients φ' and ε'_s are obtained

$$\boxed{\varphi' = \left(\frac{1}{\Phi} 1 - e^{\frac{-\Phi}{1+\Phi} \cdot \varphi}\right) \text{with } \Phi = 2\,\beta\, n\, \mu'} \tag{12}$$

$$\boxed{\varepsilon_s' = \varepsilon_s \cdot \frac{\varphi'}{\varphi}} \tag{13}$$

Here

$$\beta = \frac{x' - h'}{x'}; \qquad x' = \frac{x_0 + x_\infty}{2};$$

$$n = \frac{E_e}{E_{bo}}; \qquad \mu' = \frac{F_e'}{x' \cdot b}.$$

It is useful to calculate φ' for the individual case, since β and n may be very different. In Fig. 19 for example, the values φ' are plotted as a function of μ' for $\beta = 0.60$ and $E_{bo} = 280{,}000$ and 200,000, i.e. n = 7.5 and 10.5. The curves show that the greater the value of φ, the greater the reduction of the time-dependent deflection due to compressive reinforcement.

Comparison with a test carried out by Washa and Fluck shows that a good agreement is obtained with the aid of these approximately determined values:

Tests B 1, 2, 3 of Washa and Fluck with rectangular beam b x d = 15 x 20 cm, F_e = 2 deformed rods 16 mm diameter, concrete β_c = 210 kg/cm², E_{bo} = 185,000 kg/cm² at age of 14 days at the beginning of loading, l = 6 m, q = 160 kg/m, loading time 2½ years, $\varphi = 4.5$, ε_s = 0.70 mm/m (measured values).

The creep coefficient φ and ε_s are so large because the beams were loaded at the age of 14 days and were kept in a warm room with a relative humidity of 20 to 80%, i.e. a comparatively dry one, corresponding to the conditions in heated tall buildings.

Compressive reinforcement	F_e' =	0	0,5 F_e	F_e
	μ' =	0	1,4%	2,8%
f_0 with α_0 = 0,87	Calc.		2,6	
cm	Experiment	2,7	2,6	2,5
f_∞ with α_∞ = 1,0	Calc.	8,6	6,7	4,9
cm	Experiment	9,1	6,9	5,3

6. Closing Remark

Section 4 still requires some reworking in order to arrive at simple solutions approximating reality. It is above all important to carry out tests with continuous loading and to make observations on building constructions, so that coefficients α, α', φ and φ' etc. can be improved, especially for deformed rods.

My assistant Dipl.-Ing. Rittich helped me greatly in this work.

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Table I

Slendernesses l/h for the limiting value of the time-dependent deflection $f_{s+k} = \frac{1}{500} l$

Structures in the open $\varepsilon_s = 30 \cdot 10^{-5}$; $\varphi = 2.0$

Concrete strength	E_{b_0}	$\sigma_b =$	20	30	40	50	60	70	80	90	100	120
B 160/225	200 000 kg/cm²	$l/h =$	35	32	29	26	23	21	19			
L 300/450	280 000 kg/cm²	$l/h =$	36	34	31	29	25	24	21	20	18	16

Structures in heated buildings $\varepsilon_s = 45 \cdot 10^{-5}$; $\varphi = 4.0$

Concrete strength	E_{b_0}	$\sigma_b =$	20	30	40	50	60	70	80	90	100	120
B 160/225	200 000 kg/cm²	$l/h =$	25	22	19	17	15	13	11			
B 300/450	280 000 kg/cm²	$l/h =$	26	23	21	19	17	15	13	12	11	10

σ_b = design stress in kg/cm² for n = 15

Table II

Slenderness limits $\frac{q_D}{q}\,\frac{l_i}{h}$

Column a: for $f_s + k = \frac{1}{500}\,l$

Column b: for $f_s + k = \frac{1}{200}\,l$ or absolute limit

$\frac{q_D}{q}\,\frac{l_i}{h}$ / σ_b kg/cm²	B 160/B 225				B 300/B 450			
	In the open		In heated buildings		In the open		In heated buildings	
	a	b	a	b	a	b	a	b
20	35	50	25	50	36	50	26	50
40	29	50	19	48	31	50	21	50
60	23	50	15	36	25	50	17	41
80	19	48	11	29	21	50	13	33
100					18	47	11	28
120					16	43	10	25
130					15	41	9	24

absolute limit ⟷ $\frac{l_i}{h} \leq 35$ ⟷ $\frac{l_i}{h} \leq 50$

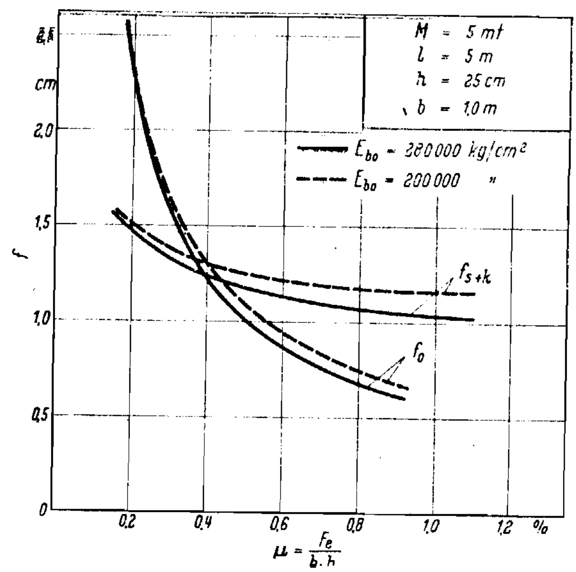


Fig. 1

Initial deflection f_o and time-dependent deflection f_{s+k} as a function of the percentage of reinforcement μ

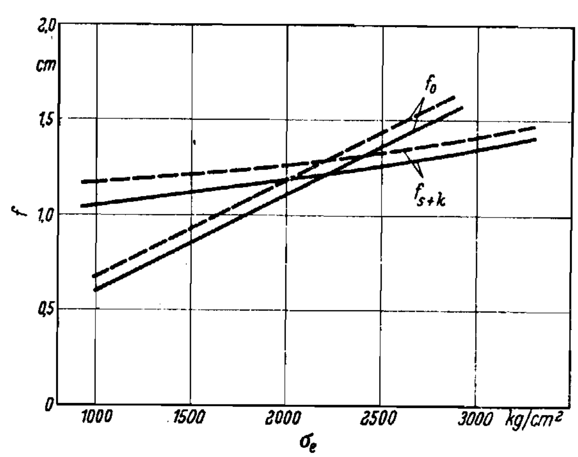


Fig. 2

Deflections as in Fig. 1, but as a function of σ_e

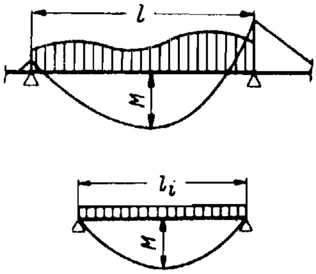


Fig. 3

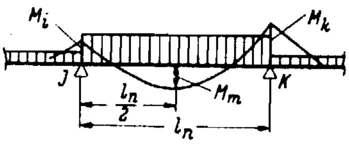


Fig. 4

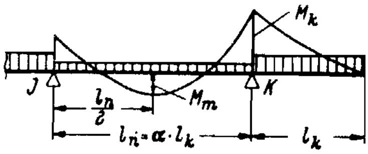


Fig. 5

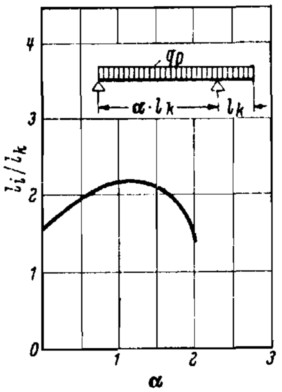


Fig. 6

Determination of $\frac{l_1}{l_k}$ as a function of α for the uniformly distributed load q_D

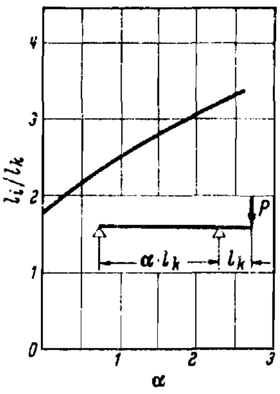
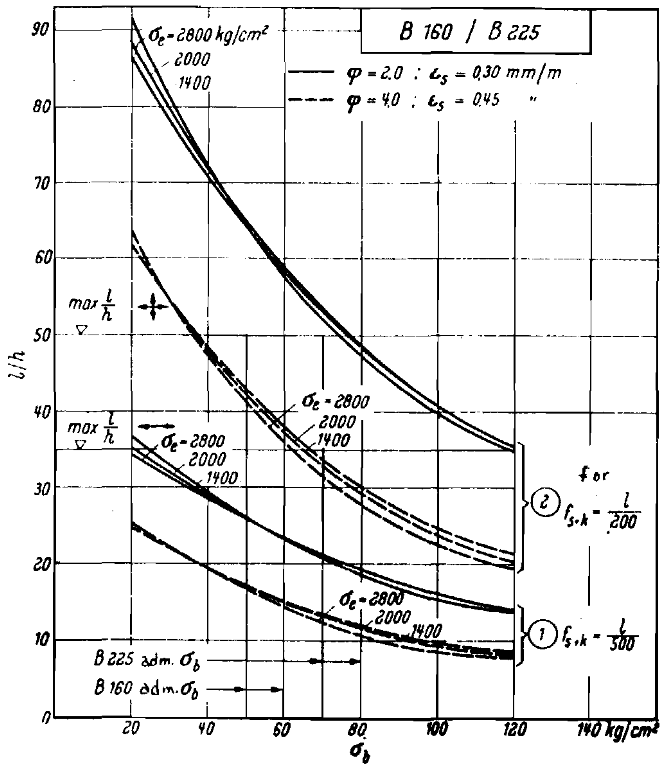


Fig. 7

Determination of $\frac{l_1}{l_k}$ as a function of α for a concentrated load P



adm. = admissible

Fig. 8

Slenderness $\frac{l}{h}$ for limiting values of the time-dependent deflection f_{s+k} for $\frac{q_D}{q} = 1$ and B160/225

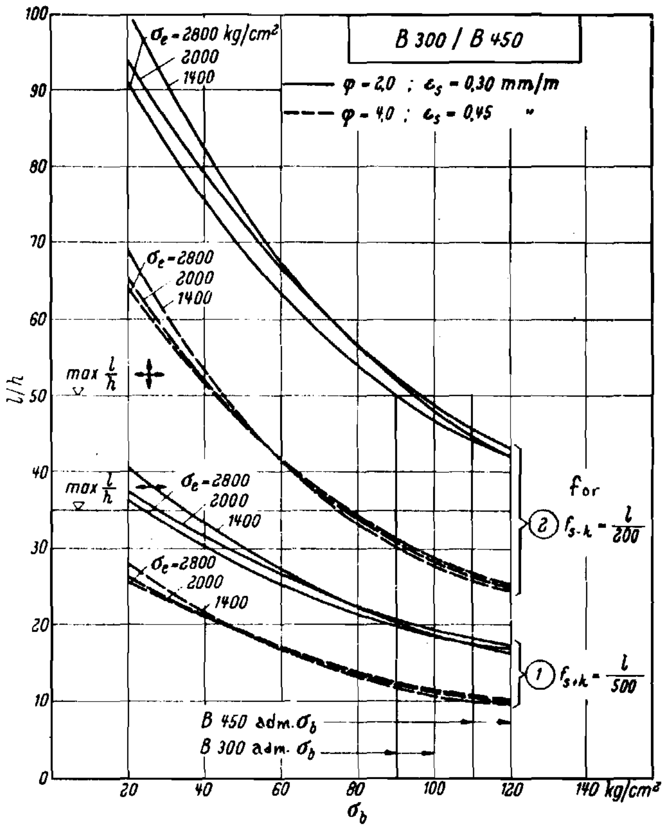


Fig. 9

Slenderness as in Fig. 8, but for B300/450

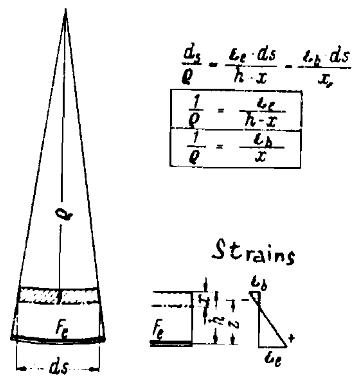


Fig. 10

Relationships between radius of curvature and strains

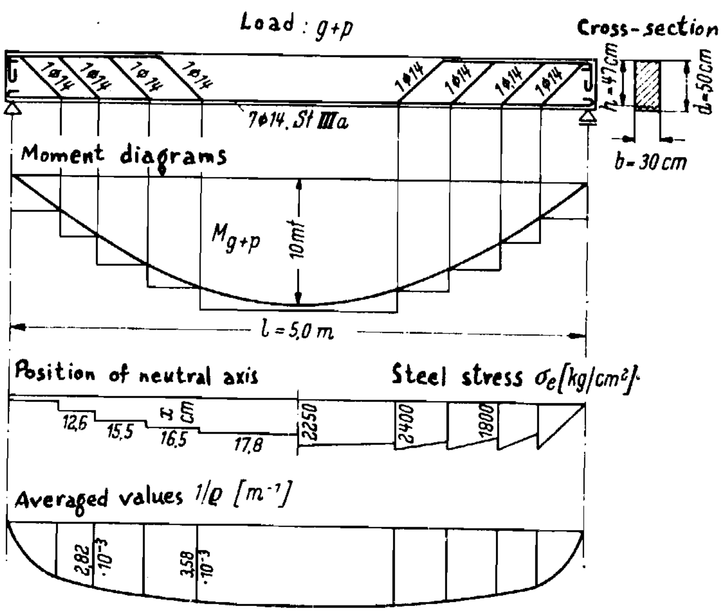


Fig. 11

Variation of the $\frac{1}{\rho}$ values for a beam on two supports with ordinary reinforcement

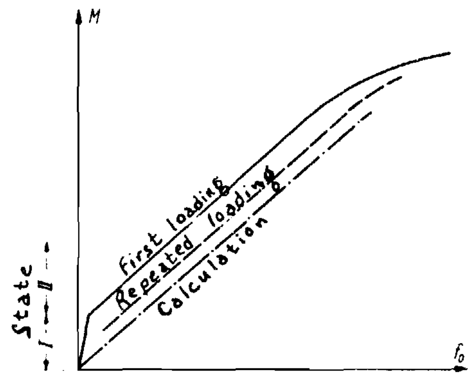


Fig. 12

Initial deflection f_0 as a function of the moments due to load

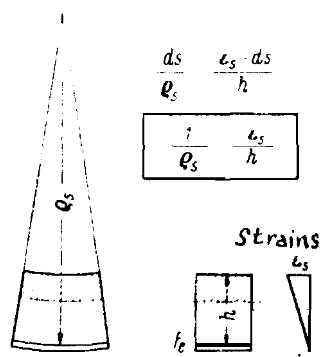


Fig. 13

Radius of curvature and elongation due to shrinkage

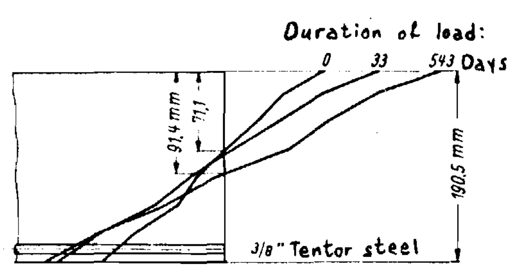


Fig. 14

Elongations in reinforced concrete beams subject to continuous loading, according to tests by Hajnal-Konyi, measured on the concrete

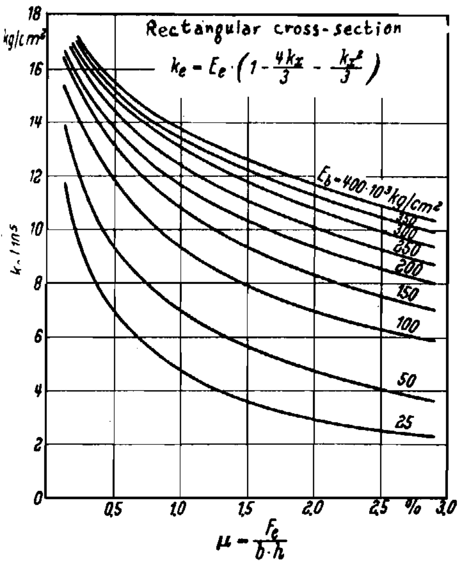


Fig. 15

The k_e values as a function of μ and E_e for rectangular cross-sections

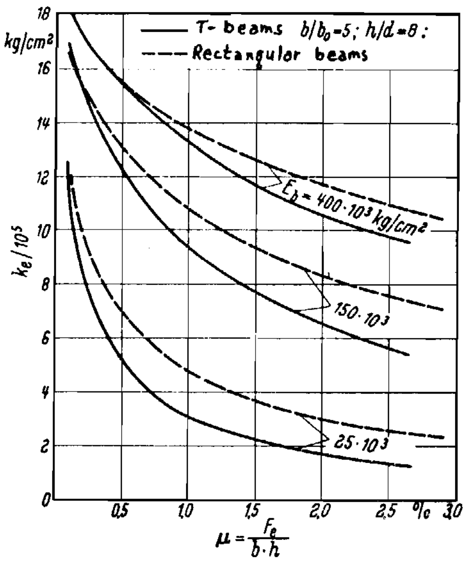


Fig. 16

k_e values as a function of μ and E_b for T-beams and rectangular cross-sections

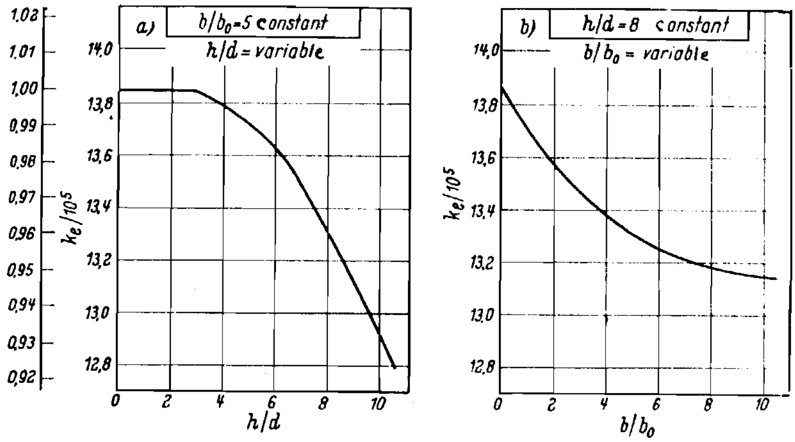


Fig. 17

Effect of cross-sectional dimensions of a T-beam on the factor k_e, for $E_b = 200 \cdot 10^3$ kg/cm^2, $\mu = 0.5\%$

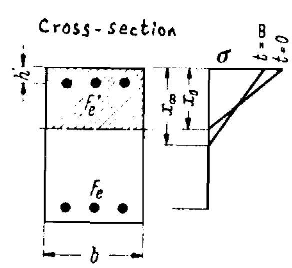


Fig. 18

Cross-section with compressive reinforcement

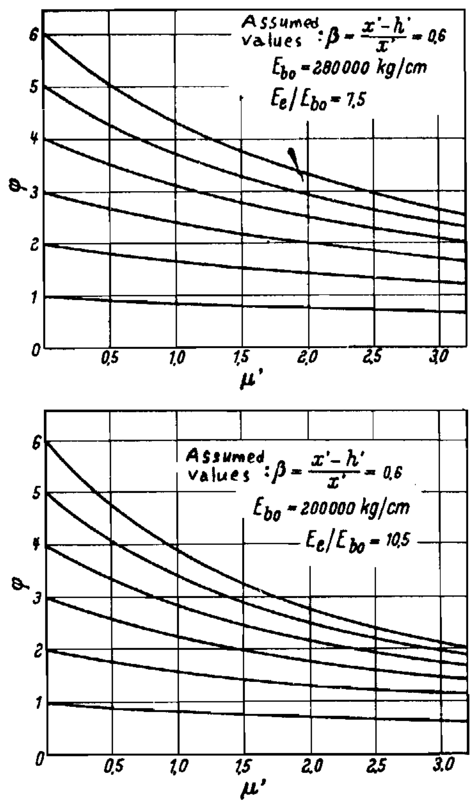


Fig. 19

Creep coefficients φ' for compressive reinforcement as a function of $\mu' = F_e/bx'$