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A GRAPHICAL SOLUTION OF THE EQUATION OF MOTION OF AN ION IN A TIME-VARYING ELECTRIC FIELD

BY

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by

P.A. Redhead

ABSTRACT

A graphical method of computing the kinematic parameters of a charged particle moving in a time-varying electric field is described. Two sets of universal curves are presented from which the above parameters may be found without recourse to any geometric constructions.

A GRAPHICAL SOLUTION OF THE EQUATION OF MOTION OF AN ION IN A TIME-VARYING ELECTRIC FIELD

INTRODUCTION

The successive integration of the equation of motion of a charged particle in a time-varying field leads to two transcendental equations, the velocity and positional equations, which are not readily solved by analytical methods. Solutions to these equations may be obtained with certain simplifying conditions. These conditions may be applied to solve the electron dynamics of velocity-modulated tubes, where the transit time is small and the initial velocity of the electrons is high with respect to any velocity changes caused by the radio-frequency field. However, when these simplifying conditions are not applicable it becomes necessary to employ graphical methods to arrive at such kinematic parameters as particle velocity, transit angle and current distribution.

Most graphical methods used for the solution of the positional and velocity equations(1)(2) are laborious and inaccurate, since it is necessary first to plot curves of a transcendental function and then to obtain values of the required parameters by geometric constructions. Curves of the function must be redrawn and the graphical construction repeated for each value of the voltage ratio, i.e., the ratio of alternating to direct voltage. The simplest of these methods is described by Kompfner(1), and an extension of Kompfner's method has been published by Garbury(3).

It was thought desirable to devise a graphical method of solution whereby a set of universal curves could be accurately drawn, and values of the required parameters could be obtained from these curves without recourse to geometric constructions. A method fulfilling these requirements is herein described*.

* Since the preparation of this paper, it has been observed that this general method has been anticipated by Guenard, Warnecke and Fauve, "Sur le rendement des tubes à modulation de vitesse" (Annales de Radioélectricité, 3, 302, 1948). The curves presented in this paper are more accurate and detailed than those in the above reference.

MOTIONAL EQUATIONS

Considering one-dimensional motion of the ion through a homogenous field represented by $E \sin \omega t$, and neglecting the effect of space charge, the equation of motion is

$$m\ddot{x} = \epsilon E \sin \omega t, \tag{1}$$

where ω is the angular frequency of the applied radio-frequency field, and all velocities are sub-relativistic.

Integrating Eq.(1), we find the velocity at time t is

$$\dot{x} = -\frac{\epsilon E}{m\omega} \cos \omega t + C. \tag{2}$$

Let $\dot{x} = v_o$, when $t = t_o$; then

$$\frac{\dot{x}}{v_o} = 1 + \frac{\epsilon E}{m v_o \omega} (\cos \omega t_o - \cos \omega t) \tag{3}$$

A second integration gives the position x as a function of time.

$$x = v_o t + \frac{\epsilon E}{m\omega^2} (\omega t \cos \omega t_o - \sin \omega t) + C_1 \tag{4}$$

Let $x = 0$, when $t = t_o$; then

$$x = v_o(t - t_o) + \frac{\epsilon E}{m\omega^2}\left[\omega(t - t_o) \cos \omega t_o - \sin \omega t + \sin \omega t_o\right] \tag{5}$$

Let the length of the radio-frequency gap be d, and $\dot{x} = v$, when $x = d$. Making the following substitutions in Eqs.(3) and (5):

a) $V_o = \frac{1}{2} m v_o^2$

b) $V = Ed$

c) $\alpha = \frac{\omega d}{v_o}$ — the unmodulated transit angle

d) $\eta = \frac{1}{2\alpha} \frac{V}{V_o}$ — the gap modulation coefficient

e) $\theta = \omega(t - t_o)$ — the transit angle

f) $\delta = \omega t_o$ — the entrance phase angle, we obtain

$$\frac{v}{v_o} = 1 + \eta \left[\cos\,\delta - \cos\,(\delta + \theta)\right] \tag{6}$$

and
$$\alpha = \theta + \eta \left[\theta\,\cos\,\delta + \sin\,\delta - \sin\,(\delta + \theta)\right] \tag{7}$$

It will be observed that the functions inside the large brackets in the above equations are functions of δ and θ only. Thus these equations may be rewritten as

$$\frac{\frac{v}{v_o} - 1}{\eta} = f_1(\delta,\theta) = \cos\,\delta - \cos\,(\delta + \theta) \tag{8}$$

and
$$\frac{\alpha - \theta}{\eta} = f_2(\delta,\theta) = \theta\,\cos\,\delta + \sin\,\delta - \sin\,(\delta + \theta) \tag{9}$$

The functions f_2 and f_1 are plotted in the δ,θ plane in Figs. 1 and 2.

METHOD OF GRAPHICAL SOLUTION

The left-hand side of Eq.(9) is a set of horizontal straight lines in the δ,θ plane. This set of lines is drawn on transparent graph paper and superimposed on the curves of f_2(Fig.1). The co-ordinates of the point of intersection of any of these lines with an f_2 contour of value equal to $\frac{a - \theta}{\eta}$ for that straight line must satisfy Eq.(9). These points of intersection are marked on the tracing paper and a curve of θ versus δ is constructed.

To obtain values of exit velocity, the curve $\theta = f(\delta)$ is superimposed on the curves of f_1 (Fig.2). The values of the f_2 contours at the points of their intersection with the $\theta = f(\delta)$ curves gives

$$f_2 = \frac{\frac{v}{v_o} - 1}{\eta}$$

as a function of δ. Since η, the modulation coefficient, is a known parameter of the system, the exit velocity v is now known for all values of the entrance phase δ.

By this method all the required kinematic parameters may be found without making any geometric constructions.

Acknowledgment

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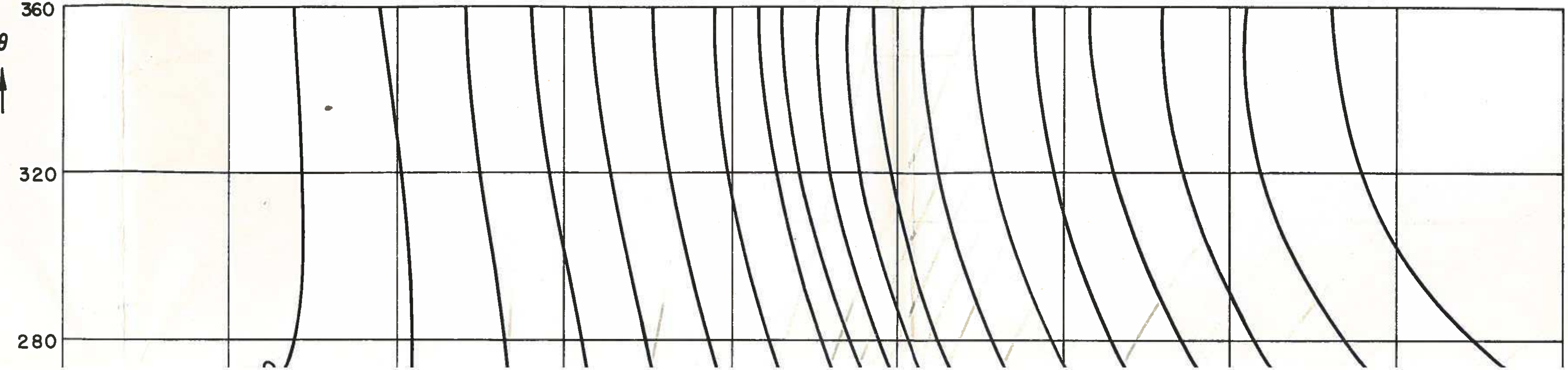
References

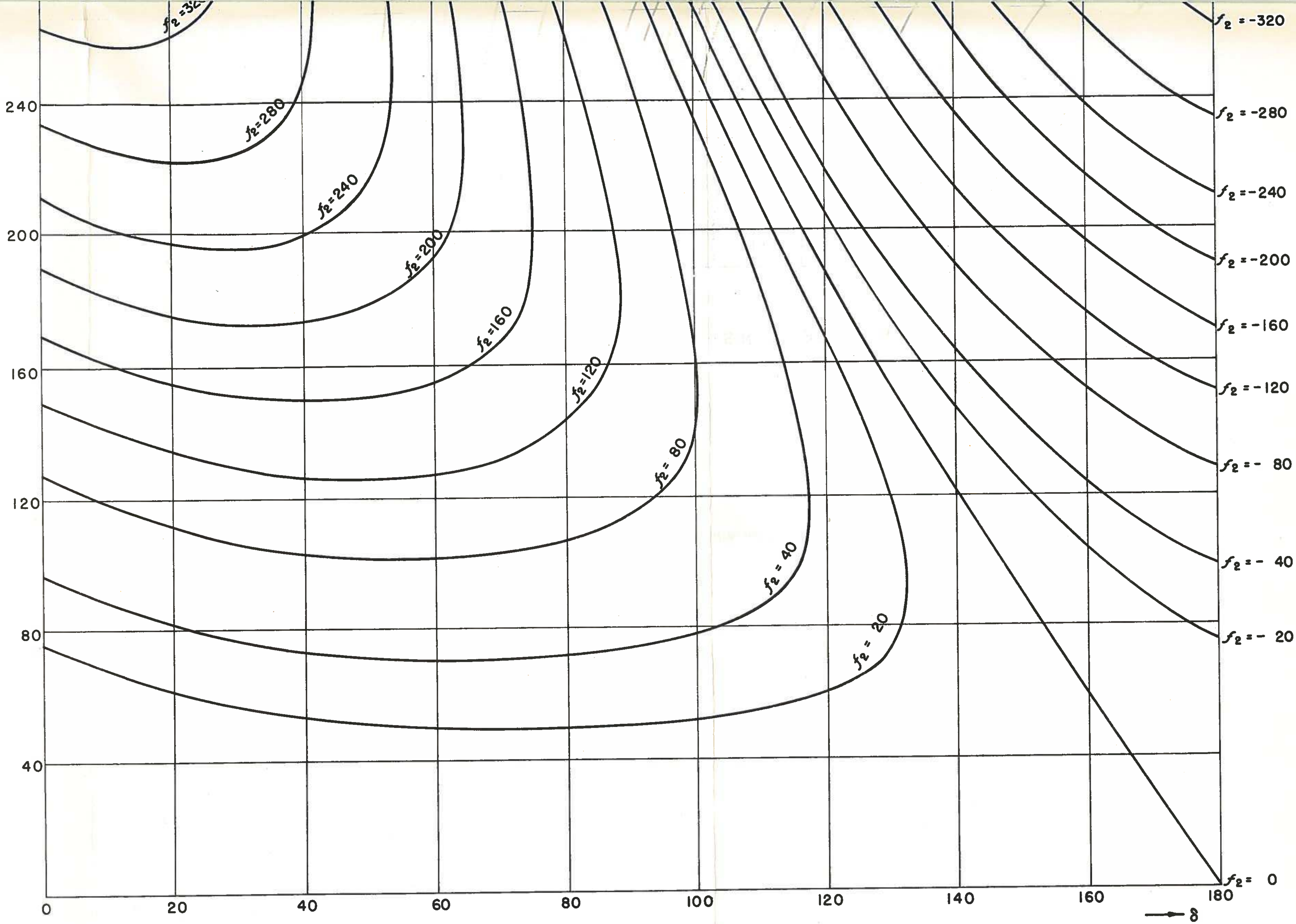
(1) R. Kompfner, Wireless Eng., 19, 2, (1942).

(2) J. Marcum, J. App. Phys., 17, 4, (1946).

(3) M. Garbury, Westinghouse Scientific Paper 1510, (1950).

$$f_2(\delta,\theta) = \theta \cos\delta + \sin\delta - \sin(\delta+\theta)$$

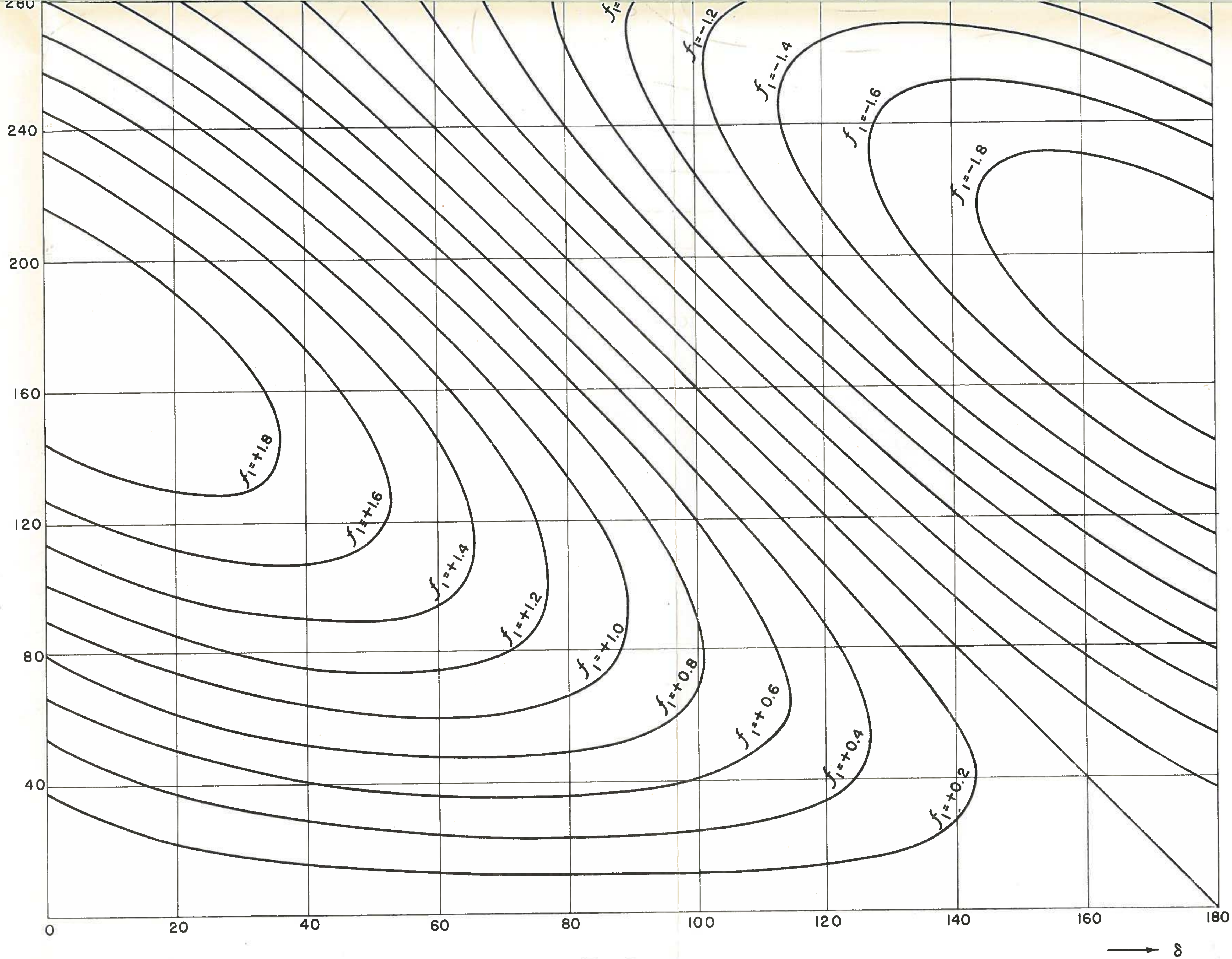




240
200
160
120
80
40
0
20
40
60
80
100
120
140
160
180
δ
$f_2 = 320$
$f_2 = 280$
$f_2 = 240$
$f_2 = 200$
$f_2 = 160$
$f_2 = 120$
$f_2 = 80$
$f_2 = 40$
$f_2 = 20$
$f_2 = -320$
$f_2 = -280$
$f_2 = -240$
$f_2 = -200$
$f_2 = -160$
$f_2 = -120$
$f_2 = -80$
$f_2 = -40$
$f_2 = -20$
$f_2 = 0$

FIG. 1

$$f_1(\delta,\theta) = \cos\delta - \cos(\delta+\theta)$$



280
240
200
160
120
80
40
0
20
40
60
80
100
120
140
160
180
δ
$f_1 = +1.8$
$f_1 = +1.6$
$f_1 = +1.4$
$f_1 = +1.2$
$f_1 = +1.0$
$f_1 = +0.8$
$f_1 = +0.6$
$f_1 = +0.4$
$f_1 = +0.2$
$f_1 = -1.2$
$f_1 = -1.4$
$f_1 = -1.6$
$f_1 = -1.8$

FIG. 2