Problems of thermal conduction in one dimension in bodies with varying conductivities by Duhamel's method

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Title: Problems of thermal conduction in one dimension in bodies with varying conductivities by Duhamel's method

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Translator: K. Shimizu

PREFACE

One of the projects of the Fire Section of the Division of Building Research concerns the development of a method for calculating the fire endurance of building elements. The main theoretical problem in this work is that the classical solutions of the Fourier equation, based on constancy of properties, cannot be used because of the wide variations in temperature involved in this experimental work.

Three papers by Masao Sawada that deal with problems of the heat conduction encountered when the heat capacity and thermal conductivity of the solid are variable, have been translated and issued as NRC Technical Translations Nos. 895, 896 and 897. From the work of Sawada it is seen that the more perfectly the mechanism of heat conduction within the solid is approached, the more limited the applicability of the method becomes, as far as the shape of the solid or the boundary conditions are concerned. Although the introduction of various numerical methods and computer techniques largely reduce the practical value of such analytical methods, there are certain fields where they are indispensable.

Ottawa,
September 1960

R.F. Legget,
Director

PROBLEMS OF THERMAL CONDUCTION IN ONE DIMENSION IN BODIES WITH VARYING CONDUCTIVITIES BY DUHAMEL'S METHOD

Abstract

In the present article, solutions are derived of problems of thermal conduction in one dimension in bodies with varying thermal conductivities by Duhamel's method in cases where the surface temperatures are controlled by periodic and non-periodic functions of time. The author wishes that the readers of the present article will refer to his article "Problems of thermal conduction in one dimension in bodies with varying conductivities" (J. Soc. Mech. Engrs., Japan, 35 (177), 10).

1. Integration by Duhamel's Method

If $\lambda = \lambda_0 r^{\mu}$, $\lambda/\overline{\rho\gamma} = a_0{}^2 r^{2-p}$, $m' = 0$ (flat slab), 1 (circular cylinder), 2 (sphere); $k = 2$ or $3/2$ (here 2), $m' + \mu - 1 = m$, $n = m/p \gtrless 0$, and further of a fractional type, the temperature is

$$v = \exp.(-i^{2k} a_0{}^2 p^2 a_s{}^2 t) \cdot J_{\pm n}(2 i^k a_s \sqrt{r^p})/(\sqrt{r^p})^n, \qquad u = \tau_1 - \tau_2/r^m \text{ (constant flow)}. \tag{1}$$

When $\lambda = \lambda_0 e^{mx}$, $\lambda/\overline{\rho\gamma} = a_0{}^2 e^{px}$, $m' = 0$, $n = m/p$.

e^x is to be substituted in (1) in place of r.

When $\lambda = \lambda_0 (c \pm r)^{\mu}$, $\lambda/\overline{\rho\gamma} = a_0{}^2 (c \pm r)^{2-p}$.

If $m' = 0$, $c \pm x$ is to be substituted in (1) in place of r. If $m' \neq 0$, and μ, p and m assume special values, a solution may be obtained if the difference between $1/r$ and $1/(c + r)$ is not too great and therefore $1/r \simeq 1/(c + r)$.

For brevity, let the double cylindrical function and the double trignometric function, which satisfy the boundary conditions, be represented by $U_n(r) = U_n(2a_s\sqrt{rp})$*. For instance, if $u = \tau_1 - \tau_2/r^m$

* Masao Sawada. J. Soc. Mech. Engrs., Japan, 35 (177), 12.

is integrated:

$$\int_{r_i}^{r_o} r^{p+m-1}[-u]\cdot U_n(r)dr = -(\tau_1 P+\tau_2 Q)/\overline{pa_s} \tag{1,}$$

where

$$P=\left| r^{p+m}U_{n+1}(r)\right|_{r_i}^{r_o}, \quad Q=\left| U_{n-1}(r)\right|_{r_i}^{r_o}.$$

P and Q will be used in what follows.

Consider a general case as a boundary condition. See Table I and Fig. 1.

$1/h_i \cdot [v']_{r_i} + b - [v]_{r_i} = 0$, $1/h_a \cdot [v']_{r_a} + [v]_{r_a} - a = 0$, where $[v]_{t=0} = f(r)$ and further $v' = \partial v/\partial r$ and the same for others.

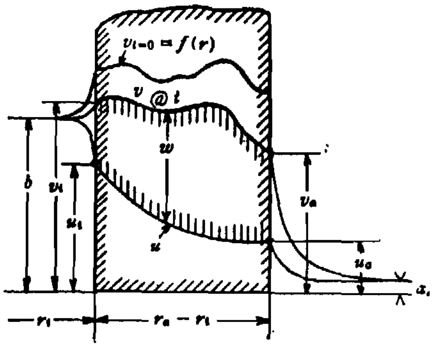

Fig. 1

Table I

	$v_{(r,t)}=u_{(r)}+w_{(r,t)}$	$u_{(r)}$ constant flow	$w_{(r,t)}$
r_i	$1/h_i\cdot[v']_{r_i}=[v]_{r_i}-b$	$1/h_i\cdot[u']_{r_i}=u_i-b$	$1/h_i\cdot[w']_{r_i}=w_i$
r_o	$1/h_o\cdot[v']_{r_o}=a-[v]_{r_o}$	$1/h_o\cdot[u']_{r_o}=a-u_o$	$1/h_o\cdot[w']_{r_o}=-w_o$
t	$[v]_{t=0}=f(r)$	$u=[v]_{t=\infty}$	$[w]_{t=0}=f(r)-u$

If $a = \psi(t)$, and $b = \phi(t)$, $[w]_{t=0} = 0 \therefore u = f(r)$.

If $1/h_a = 0$ and $1/h_i = 0$, $[v]_{r_a} = a$ or $\psi(t)$; $[v]_{r_i} = b$ or $\phi(t)$, respectively.

If $h_a = 0$ and $h_i = 0$, $[v']_{r_a} = 0$, $[v']_{r_i} = 0$, respectively.

Further, when a and b are functions of t, a solution is obtained for the case when they are always constants. The purpose of the present article is to apply Duhamel's method* to the solution so obtained.

Generally, in order to solve cases where surrounding temperature is a function of time, in addition to the above, other methods, such as that of ordinary integration, Riemann, Stokes, Angstrom*, Carslaw* and Bromwich** are available. However, it seems to the

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* Carslaw. Conduction of heat. pp. 17, 68, 42 and 210-224.

** Bromwich. Phil. Mag. v. 37, 1919. J. Soc. Mech. Engrs., Japan, 36 (189).

author that Duhamel's method is the most convenient when the initial temperature inside the body is a function of dimension. The calculation is shown below for convenience:

(i) When the change is periodic.* Let $\beta_s = -(a_0 p a_s)^2$.

$$a=\psi(t)=\psi_0+\sum_{k=1}^{\infty}[\psi_k' \cos k\omega t+\psi_k'' \sin k\omega t], \qquad b=\phi(t)=\phi_0+\sum_{k=1}^{\infty}[\phi_k' \cos k\omega t+\phi_k'' \sin k\omega t]$$

$$\int_0^t \psi(\sigma)\cdot\partial_t e^{\beta_s(t-\sigma)}d\sigma=\beta_s G_k(t) \quad \text{......}(1_2)$$

$$\int_0^t \psi(\sigma)\cdot\partial_t e^{\beta_s(t-\sigma)}d\sigma=\beta_s H_k(t) \quad \text{......}(1_3)$$

where $G_k(t)=\psi_0(e^{\beta_s t}-1)/\beta_s+\xi_{\psi,k}\{[\beta_s\cdot\sin(k\omega t+\theta_{\psi,k})+k\omega\cdot\cos(k\omega t+\theta_{\psi,k})] + e^{\beta_s t}[\beta_s\cdot\sin\theta_{\psi,k}+k\omega\cdot\cos\theta_{\psi,k}]\}/\overline{\beta_s^2+(k\omega)^2}$,

$H_k(t)=\phi_0(e^{\beta_s t}-1)/\beta_s+\xi_{\phi,k}\{[\beta_s\cdot\sin(k\omega t+\theta_{\phi,k})+k\omega\cdot\cos(k\omega t+\theta_{\phi,k})] + e^{\beta_s t}[\beta_s\cdot\sin\theta_{\phi,k}+k\omega\cdot\cos\theta_{\phi,k}]\}/\overline{\beta_s^2+(k\omega)^2}$

$\xi_{\psi,k}=(\psi_k'^2+\psi_k''^2)^{1/2}, \quad \theta_{\psi,k}=\tan^{-1}\psi_k''/\psi_k', \quad \xi_{\phi,k}=(\phi_k'^2+\phi_k''^2)^{1/2}, \quad \theta_{\phi,k}=\tan^{-1}\phi_k''/\phi_k'$

(ii) When the change is non-periodic. β_s the same as before.

$$a=\psi(t)=\psi_k(t)=\psi_0+\sum_{k=1}^{k}\psi_k\cdot t^k, \qquad b=\phi(t)=\phi_k(t)=\phi_0+\sum_{k=1}^{k}\phi_k\cdot t^k$$

$$\int_0^t \psi_k(\sigma)\cdot\partial_t e^{\beta_s(t-\sigma)}d\sigma=\beta_s\cdot\Psi_k\cdot e^{\beta_s t}-\Psi_k(t) \quad \text{......}(1_4)$$

$$\int_0^t \phi_k(\sigma)\cdot\partial_t e^{\beta_s(t-\sigma)}d\sigma=\beta_s\cdot\Phi_k\cdot e^{\beta_s t}-\Phi_k(t) \quad \text{......}(1_5)$$

where $\Psi_k=\psi_0/\beta_s+\sum_{k=1}^{k}k!\cdot\psi_k/\beta_s^{k+1}, \quad \Psi_k(t)=\psi_k(t)+\psi_{k-1}(t)/\beta_s+\psi_{k-2}(t)/\beta_s^2+\ldots\ldots+\psi_1(t)/\beta_s^{k-1}+k!\cdot\psi_k/\beta_s^k$,

$\Phi_k=\phi_0/\beta_s+\sum_{k=1}^{k}k!\cdot\phi_k/\beta_s^{k+1}, \quad \Phi_k(t)=\phi_k(t)+\phi_{k-1}(t)/\beta_s+\phi_{k-2}(t)/\beta_s^2+\ldots\ldots+\phi_1(t)/\beta_s^{k-1}+k!\cdot\phi_k/\beta_s^k$,

$\psi_k(t)=\psi_0+\sum_{k=1}^{k}\psi_k\cdot t^k, \quad \psi_{k-1}(t)=\sum_{k=1}^{k}k\cdot\psi_k\cdot t^{k-1}, \quad \psi_{k-2}(t)=\sum_{k=2}^{k}k!/(k-2)!\cdot\psi_k\cdot t^{k-2}, \quad \psi_{k-3}(t)=\sum_{k=3}^{k}k!/(k-3)!\cdot\psi_k\cdot t^{k-3}$

$\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots,$

$\psi_2(t)=(k-2)!\cdot\psi_{k-2}+(k-1)!\cdot\psi_{k-1}\cdot t+k!/2!\cdot\psi_k\cdot t^2, \quad \psi_1(t)=(k-1)!\psi_{k-1}+k!\psi_k\cdot t, \ldots\ldots$

$\phi_{k-1}(t)$, $\phi_{k-2}(t)$, ...$\phi_1(t)$ take the same form as $\psi_{k-1}(t)$, $\psi_{k-2}(t)$, ... $\psi_1(t)$. In what follows, references will be made to $G_k(t)$, $H_k(t)$, $\Psi_k(t)$, Ψ_k, $\Phi_k(t)$ and Φ_k.

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* Kumabe, Kazuo: J. Soc. Mech. Engrs., Japan, 28 (104), 976.

2. Problems of Thermal Conduction

[1°] $1/h_1 \cdot [\partial v/\partial r]_{r_1} = [v]_{r_1} - b, \quad 1/h_a \cdot [\partial v/\partial r]_{r_a} = a - [v]_{r_a}$,

and further $[v]_{t=0} = f(r)$.

(I) When a and b are fixed.

Solution:
$$v = u + p\sum_{s=1}^{\infty}\mathfrak{B}_s \cdot a_s^2 \cdot \exp.(-a_0^2 p^2 a_s^2 t)\, U_n(r)\int_{r_i}^{r_a} r^{p+m-1}[f(r)-u]\cdot U_n(r)dr \quad \cdots\cdots(2)$$

where
$$u = \tau_1 - \tau_2/r^m, \qquad \tau_1 = [b(m/\overline{h_a r_a^{m+1}} - r_a^{-m}) + a(m/\overline{h_i r_i^{m+1}} + r_i^{-m})]/q,$$
$$\tau_2 = (a-b)/q, \qquad q = m(1/\overline{h_a r_a^{m+1}} + 1/\overline{h_i r_i^{m+1}} + (r_i^{-m} - r_a^{-m}),$$
$$U_n(r) = \{J_n(r)\cdot[\text{No. }1] - J_{-n}(r)\cdot[\text{No. }2]\}/\sqrt{r^m}, \qquad [\text{No. }1]^{(*)} = -[p(r_i)/2r_i h_i]J_{-(n+1)}(r_i) + J_{-n}(r_i),$$
$[\text{No. }2]^{(*)} = [p(r_i)/2r_i h_i]\cdot J_{n+1}(r_i) + J_n(r_i)$, a_s is a positive root of $\overline{[U_n'(r)]_{r_a} + h_a U_n(r_a) = 0}$
$$\mathfrak{B}_s = \{[(pa_s)^2 r_a^p + h_a^2 r_a^2 - m h_a r_a] r_a^m [U_n(r_a)]^2 - [(pa_s)^2 r_i^{p-2} - m h_i/r_i + h_i^2](\sin n\pi/\pi)^2\}^{-1}$$
$$\partial v/\partial r = m\tau_2/r^{m+1} + p^2\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^3 r^{p-1}\cdot\exp.(-a_0^2 p^2 a_s^2 t)\cdot U_{n+1}(r)\cdot\left\{\int_{r_i}^{r_a} r^{p+m-1} f(r)\cdot U_n(r)dr - (\tau_1 P + \tau_2 Q)/\overline{pa_s}\right\} \cdots\cdots(2_1)$$

Thus the quantity of heat flow can be calculated.

(II) When a and b are periodic functions. Refer to 1, (i).

Solution:
$$v = (a_0 p)^2\sum_{s=1}^{\infty}\cdot\sum_{k=1}^{\infty}\mathfrak{B}_s\cdot a_s^3\cdot U_n(r)\{[X][G(t)] + [Y][H(t)]\} + [Z] \quad \cdots\cdots(2_3)$$

where
$$X = [(m/\overline{h_a r_a^{m+1}} - r_a^{-m})P - Q]/q, \qquad Y = [(m/\overline{h_i r_i^{m+1}} + r_i^{-m})P + Q]/q,$$
$$Z = p\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot\exp.(-a_0^2 p^2 a_s^2 t)\cdot U_n(r)\int_{r_i}^{r_a} r^{p+m-1}\cdot f(r)\cdot U_n(r)dr.$$

(III) When a and b are non-periodic functions. Refer to 1, (ii).

Solution: $v = \sum_{s=1}^{\infty}\cdot\sum_{k=1}^{\infty}\mathfrak{B}_s\cdot a_s\cdot U_n(r)\left\{[X][\Phi_k(t)] + [Y][\Psi_k(t)] + (a_0 p a_s)^2 e^{-a_0^2 p^2 a_s^2 t}[[X][\Phi_k] + [Y][\Psi_k]]\right\} + [Z] \quad \ldots(2_4)$

In (II) and (III), the effect of $\exp.(-a_0^2 p^2 a_s^2 t)$ will diminish with time and finally become non-existent. The case will be further simplified if $f(r) = 0$ or a constant.

[2°] $[v]_{r_1} = b, \quad 1/h_a \cdot [\partial v/\partial r]_{r_a} = a - [v]_{r_a}, \quad [v]_{t=0} = f(r)$.

(I) When a and b are constants.

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* Sawada, Masao: J. Soc. Mech. Engrs., Japan, 35 (177), 14 with revisions as in the present article.

Solution: $$v=u+p^3\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^2\cdot\exp.(-a_0^2p^2a_s^2t)\cdot U_n(r)\left\{\int_{r_i}^{r_a}r^{p+m-1}f(r)\cdot U_n(r)dr-(\tau_1P+\tau_2Q)/\overline{pa_s}\right\} \quad (2_5)$$

where $u=\tau_1-\tau_2/r^m,\quad \tau_1=[b(m/\overline{h_ar_a}^{m+1}-r_a^{-m})+ar_i^{-m}]/q,\quad \tau_2=(a-b)/q,\quad q=m/\overline{h_ar_a}^{m+1}+(r_i^{-m}-r_a^{-m}),$

$U_n(r)=[J_n(r)\cdot J_{-n}(r_i)\cdot J_{-n}(r)\cdot J_n(r_i)]/\sqrt{r^m}$, a_s is a positive root of $[U_n'(r)]_{r_a}+h_aU_n(r_a)=0$. $\mathfrak{B}_s=\left\{[pa_s]^2r_a^p+h_a^2r_a^2-mh_ar_a]r_a^m[U_n(r_a)]^2-(p\sin n\pi/\pi)^2\right\}^{-1}$.

Further $$\partial v/\partial r=m\tau_2/r^{m+1}+p^4\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^3\cdot r^{p-1}\cdot U_{n+1}(r)\cdot\exp.(-a_0^2p^2a_s^2t)\cdot\int_{r_i}^{r_a}r^{p+m-1}[f(r)-u]U_n(r)dr \quad (2_6)$$

(II) a and b are periodic functions. Refer to 1, (i).

Solution: $$v=a_0^2p^4\sum_{s=1}^{\infty}\cdot\sum_{k=1}^{\infty}\mathfrak{B}_s\cdot a_s^3\cdot U_n(r)\left\{[X][(G(t)]+[Y][H(t)]\right\}+[Z] \quad (2_7)$$

where $X=[(m/\overline{h_ar_a}^{m+2}-r_a^{-m})P-Q]/q,\qquad Y=[r_i^{-m}P+Q]/q,$

$Z=p^3\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^2\cdot\exp.(-a_0^2p^2a_s^2t)\cdot U_n(r)\int_{r_i}^{r_a}r^{p+m-1}f(r)\cdot U_n(r)dr$

(III) a and b are non-periodic functions. Refer to 1, (ii).

Solution: $$v=p^2\sum_{s=1}^{\infty}\cdot\sum_{k=1}^{\infty}\mathfrak{B}_s\cdot a_s\cdot U_n(r)\left\{[X][\Phi_k(t)+(a_0pa_s)^2\cdot\Phi_k\cdot\exp.(-a_0^2p^2a_s^2t)]\right.$$
$$\left.+[Y][\Psi_k(t)+(a_0pa_s)^2\cdot\Psi_k\cdot\exp.(-a_0^2p^2a_s^2t)]\right\}+[Z] \quad (2_8)$$

[3°] $[v]_{r_i}=b,\qquad [v]_{r_a}=a,\qquad [v]_{t=0}=f(r)$

(I) When a and b are constants.

Solution: $$v=u+p\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^2\cdot\exp.(-a_0^2p^2a_s^2t)\cdot U_n(r)\int_{r}^{r_a}r^{p+m-1}[f(r)-u]U_n(r)dr \quad (2_9)$$

where $u=\tau_1-\tau_2/r^m,\quad \tau_1=(ar_i^{-m}-br_a^{-m})/q,\quad \tau_2=(a-b)/q,\quad q=r_i^{-m}-r_a^{-m},$

$U_n(r)=[J_n(r)\cdot J_{-n}(r_i)-J_{-n}(r)\cdot J_n(r_i)]/\sqrt{r^m}$, a_s is a positive root of $U_n(r_a)=0$

$\mathfrak{B}_s=(\pi/\sin n\pi)^2/(\theta_s^2-1),\qquad \theta_s=J_n(r_a)/J_n(r_i)=J_{-n}(r_a)/J_{-n}(r_i).$

Further $$\partial v/\partial r=m\tau_2/r^{m+1}+p^2\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^3\cdot\exp.(-a_0^2p^2a_s^2t)\cdot U_{n+1}(r)\cdot r^{p-1}$$
$$\times\left\{\int_{r_i}^{r_a}r^{p+m-1}f(r)\cdot U_n(r)dr-(\tau_1P+\tau_2Q)/\overline{pa_s}\right\} \quad (2_{10})$$

(II) a and b are periodic functions. Refer to 1, (i).

Solution:

$$v=(a_0p)^2\sum_{s=1}^{\infty}\cdot\sum_{k=1}^{\infty}\mathfrak{B}_s\cdot a_s^3\cdot U_n(r)\left\{[X][G(t)]+[Y][H(t)]\right\}+[Z] \quad (2_{11})$$

$X=-(r_a^{-m}P+Q)/q,\qquad Y=(r_i^{-m}P+Q)/q,$

$Z=p\sum_{s=1}^{\infty}\mathfrak{B}_s\cdot a_s^2\cdot\exp.(-a_0^2p^2a_s^2t)\cdot U_n(r)\int_{r}^{r_a}r^{p+m-1}f(r)\cdot U_n(r)dr$

(III) a and b are non-periodic functions. See 1, (ii).

Solution:

$$v=\sum_{s=1}^{\infty}\cdot\sum_{k=1}^{\infty}\mathfrak{B}_s\cdot a_s\cdot U_n(r)\Big\{[X][\Phi_k(t)+(a_0pa_s)^2\cdot\Phi_k\cdot\exp.(-a_0^2p^2a_s^2t)] + [Y][\Psi_k(t)+(a_0pa_s)^2\cdot\Psi_k\cdot\exp.(-a_0^2p^2a_s^2t)]\Big\}+[Z] \quad \ldots\ldots\ldots (2_{12})$$

In the preceding sections, the author dealt with the case of cooling with the boundary condition $1/h_1 \cdot [v']_{r_1} = [v]_{r_1} - b$, $1/h_a \cdot [v']_{r_a} = a - [v]_{r_x}$ with particular reference to 3 cases, $h_1 = 0$, $1/h_1 = 1/h_a = 0$. There are cases in which $h_1 = 0$ and $h_1 = h_a = 0$. Since solutions are readily available in these cases, they will not be dealt with in the present paper.

Generally, the consideration of the case of heating or cooling gives rise to the following:

$$\pm 1/h_i\cdot[v']_{r_i}=[v]_{r_i}-b, \qquad \pm 1/h_a\cdot[v']_{r_a}=a-[v]_{r_a}$$

where $h_1 = 0$, finite or infinite; $h_a = 0$, finite or infinite;
$b = 0$, constant or $\phi(t)$; $a = 0$, constant or $\psi(t)$;
$b = a$ or $b \neq a$; $[v]_{t=0} = 0$, constant or $f(r)$.

Further, the order n of the function $U_n(r)$ of the solution can be an integer or a fraction, positive or negative. Thus, various combinations of these possibilities give rise to varied cases.
In the present article solutions were obtained by Duhamel's method only. However, the author wishes to mention that, when $\pm 1/h_a [v']_{r_a} = a - [v]_{r_a}$; $a = 0$, constant or $\psi(t)$; $[v]_{t=0} = 0$, a solution can be obtained, as in the case of homogeneous bodies, by Bromwich's method.

August 15, 1932