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Laminar flow in channels with porous walls, revisited☆

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Abstract

An analytical solution for the pressure drop in fluid flow in a rectangular slit and cylindrical tube with porous walls is presented for the case of constant wall permeability. It is shown that model predictions for the particular case of constant wall velocity for rectangular slit with permeable walls agrees very well with Berman's solution [J. Appl. Phys. 24 (1953) 1232]. The derivation presented in this work leads to analytical expressions for pressure drop as a function of wall permeability, channel dimension, axial position and fluid properties. These analytical expressions for constant wall permeability could be used to benchmark numerical routines for fluid flow modeling past semi-permeable membranes and for quick engineering estimates for pressure drop in cross-flow membrane modules. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Theory; Fluid flow; Porous slit; Porous tube

1. Introduction

In his classical paper of 1953, Berman [1] developed solutions to the Navier–Stokes equations for fluid flow in a rectangular slit with two equally porous walls. The solution derived from perturbation methods was based on the following assumptions: "(1) a steady state prevails; (2) the fluid is incompressible; (3) no external forces act on the fluid; (4) the flow is laminar and (5) the velocity of the fluid leaving the walls of the channel is independent of position" [1]. A similar perturbation approach was used by Yuan and Finkelstein [2] in 1956 to extend Berman's work to a cylindrical geometry with uniform suction/injection along the length of the porous tube. Kozinski et al. [3] developed analytical expressions for pressure drop through porous slit and tube for the case of an arbitrary exponential dependence of the wall velocity on the slit/tube length.

Over the years, several authors have used Berman's [1] solution to benchmark their numerical routines or mathematical models for predicting pressure drop in the feed channel of a semi-permeable membrane [4–9]. In membrane transport, however, the local permeate velocity is a function of the local fluid pressure. An analytical expression for the pressure drop without assumption (5) above is, therefore, desirable.

In this short note, an analytical solution is presented where assumption (5) above, is relaxed by assuming the wall velocity to be proportional to the local trans-membrane pressure difference; the proportionality constant being the membrane permeability. A comparison with the particular case of a constant wall velocity shows this solution to agree very well with Berman's solution [1]. Expression for pressure loss through a porous tube with the wall velocity

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Nomenclature

A	wall (membrane) permeability (m/s Pa)
f	fractional recovery (Q_p/Q_i)
h	half slit height (m)
L	slit (or tube) length (m)
N_{Re}	inlet Reynolds number
$N_{Re\mathrm{w}}$	Reynolds number based on wall velocity
P	pressure (Pa)
ΔP	pressure drop ($P_i - P$) (Pa)
Q	flow rate (m^3/s)
R	tube radius (m)
$\bar{u}(0)$	average inlet velocity (m/s)
v_w	wall velocity (m/s)
W	slit width (m)
z	axial coordinate in the direction of fluid flow (m)

Greek letters

λ	constant defined in the text (1/m)
μ	viscosity (in this work $= 10^{-3}$) (Pa s)
ρ	density (in this work $= 10^3$) (kg/m^3)

Subscripts

i	inlet
p	permeate

being proportional to the trans-membrane pressure difference is also presented.

2. Model development for slit flow

Consider a rectangular slit with width, W and slit height of $2h$. Let the z-coordinate denote the direction of fluid flow with $z = 0$ denoting the inlet to the slit and $z = L$ denoting the exit from the slit. For slit flow with impermeable walls, the flow rate through the slit is related to the slit pressure gradient by [10]

$$Q = \frac{2}{3}\frac{h^3 W}{\mu}\left(-\frac{\mathrm{d}P}{\mathrm{d}z}\right) \tag{1}$$

where Q is the flow rate through the slit, P the pressure and m the fluid viscosity. For a slit with impermeable walls, Q is independent of z and therefore the pressure gradient is also constant leading to a linear decay of pressure along the slit length. Eq. (1) is derived from the Navier–Stokes equations of motion and the equation of continuity by assuming that the predominant pressure gradient is in the z-direction and that the only non-zero component of velocity is also in the z-direction. For a permeable wall, the volumetric flow rate Q would change along the channel length due to permeation through the walls.

Slattery [11] has presented a development for pressure drop though a conical section where the cross-section of the cone varies as a function of the axial length. The Hagan–Poiseuille equation [10] for tube flow (constant cross-section) was differentiated and applied locally to infinitesimal sections of the cone, assuming that within that small section, the cross-section was constant. The result was then integrated along the axial length of the cone to estimate the overall pressure drop. A similar procedure can be followed for permeable walls as follows.

Differentiating Eq. (1) with z, we get

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = \frac{2}{3}\frac{h^3 W}{\mu}\left(-\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}\right) = -2v_w W \tag{2}$$

where v_w is velocity of the fluid leaving through the two permeable walls of the slit. Eq. (2) relates the differential change in pressure drop in an infinitesimal section dz of the slit due to loss of fluid through the permeable walls. Since it is a differential form of Eq. (1), it also assumes that in the infinitesimal section dz, the predominant pressure gradient is still in the z-direction and that the fractional flow rate out of the section dz is negligible with respect to the (local) axial flow rate, Q.

The boundary conditions for Eq. (2) are

$$z = 0, \quad P = P_i \tag{3a}$$

$$z = 0, \quad \left(-\frac{\mathrm{d}P}{\mathrm{d}z}\right) = \frac{3}{2}\frac{\mu}{h^3 W}Q_i \tag{3b}$$

where Q_i is the fluid flow rate at the inlet of the slit.

For the case of constant wall velocity, Eq. (2) can be integrated using Eqs. (3a) and (3b) to give

$$P_i - P = \Delta P = \frac{3}{2}\frac{\mu}{h^3 W}Q_i z\left(1 - \frac{2v_w W z}{2Q_i}\right) \tag{4}$$

where ΔP is the pressure drop at any z-location along the slit. Setting $z = L$, the overall slit pressure drop can be calculated as

$$P_{\rm i} - P_{\rm o} = \Delta P = \frac{3}{2}\frac{\mu}{h^3 W} Q_{\rm i} L \left(1 - \frac{2v_{\rm w} W L}{2Q_{\rm i}}\right) = (\Delta P)_{v_{\rm w}=0}\left(1 - \frac{f}{2}\right) \quad (5)$$

where $P_{\rm o}$ is the exit pressure of the fluid, $(\Delta P)_{v_{\rm w}=0}$ the pressure drop for a slit with impermeable walls and f is the fractional permeate recovery (total permeate flow rate/inlet flow rate) for the slit of length L.

Berman [1] derived the following expression for the pressure drop for fluid flow through a slit with permeable walls (constant wall velocity)

$$\Delta P = \left(\frac{1}{2}\rho \bar{u}^2(0)\right)\left(\frac{24}{N_{Re}} - \frac{648}{35}\frac{N_{Re\rm w}}{N_{Re}}\right) \times \left(1 - \frac{2N_{Re\rm w}}{N_{Re}}\frac{z}{h}\right)\left(\frac{z}{h}\right) \quad (6)$$

where r is the fluid density and $\bar{u}(0)$ the average inlet velocity. The quantities N_{Re} and $N_{Re\rm w}$ are the inlet and "wall" Reynolds number, respectively, given by

$$N_{Re} = \frac{4h\bar{u}(0)\mu}{\rho} \quad (7a)$$

$$N_{Re\rm w} = \frac{h v_{\rm w}\mu}{\rho} \quad (7b)$$

It will be shown later that predictions using Eq. (4) agree very well with Eq. (6) derived by Berman [1] using a perturbation solution to the two-dimensional Navier–Stokes equation.

For the case of wall velocity being proportional to the trans-membrane pressure difference (constant wall permeability), Eq. (2) becomes

$$\frac{2}{3}\frac{h^3 W}{\mu}\left(-\frac{{\rm d}^2 P}{{\rm d}z^2}\right) = -2v_{\rm w}W = -2A(P - P_{\rm p})W \quad (8)$$

where $P_{\rm p}$ is the permeate side pressure (usually atmospheric) and A the membrane permeability. The quantity $v_{\rm w}$ is assumed to be proportional to the trans-membrane pressure difference $(P - P_{\rm p})$. Defining $P^* = (P - P_{\rm p})$ and re-arranging Eq. (8), we get

$$\frac{{\rm d}^2 P^*}{{\rm d}z^2} = \frac{3\mu A}{h^3} P^* \quad (9)$$

Eq. (9) can be solved along with Eqs. (3a) and (3b) to give the expression for pressure drop in a slit with constant wall permeability

$$P_{\rm i} - P = \Delta P = \frac{1}{2} \times \frac{3}{2}\frac{\mu}{h^3 W}\frac{Q_{\rm i}}{\lambda}({\rm e}^{\lambda z} - {\rm e}^{-\lambda z}) + (P_{\rm i} - P_{\rm p})\left(1 - \frac{{\rm e}^{\lambda z} + {\rm e}^{-\lambda z}}{2}\right) \quad (10)$$

where

$$\lambda = +\sqrt{\frac{3\mu A}{h^3}} \quad (11)$$

The fractional recovery is given by

$$f = \frac{Q_{\rm p}}{Q_{\rm i}} = \frac{2WA\int_0^L (P - P_{\rm p})\,{\rm d}z}{Q_{\rm i}} = \frac{WA}{Q_{\rm i}}\left\{(P_{\rm i} - P_{\rm p})({\rm e}^{\lambda z} - {\rm e}^{-\lambda z}) - \frac{3}{2}\frac{\mu}{h^3 W}\frac{Q_{\rm i}}{\lambda}({\rm e}^{\lambda z} + {\rm e}^{-\lambda z} - 2)\right\} \quad (12)$$

3. Model development for tube flow

Consider a cylindrical tube with radius R and the z-coordinate denoting direction of fluid flow as in the case above. Starting with the Hagan–Poiseuille result for impermeable tube wall [10], and following the procedure outlined above, the following expressions for pressure drop can be derived.

For constant wall velocity:

$$\Delta P = \frac{8\mu Q_{\rm i} L}{\pi R^4}\left(1 - \frac{2\pi R L v_{\rm w}}{2Q_{\rm i}}\right) = (\Delta P)_{v_{\rm w}=0}\left(1 - \frac{f}{2}\right) \quad (13)$$

For constant wall permeability:

$$\Delta P = \frac{1}{2}\frac{8\mu Q_{\rm i}}{\pi R^4 \lambda}({\rm e}^{\lambda z} - {\rm e}^{-\lambda z}) + (P_{\rm i} - P_{\rm p})\left(1 - \frac{{\rm e}^{\lambda z} + {\rm e}^{-\lambda z}}{2}\right) \quad (14)$$

where

$$\lambda = +\sqrt{\frac{16\mu A}{R^3}} \quad (15)$$

4. Discussion and conclusions

It should be noted that each of the expressions for pressure drop in permeable slit/tube simplifies to known expressions for slit/tube flow with impermeable walls by setting $v_w = 0$ for the case of constant wall velocity and $\lambda = 0$ for the case of wall velocity proportional to the local trans-membrane pressure difference. It should also be emphasized that the applicability of Eq. (10) for slit flow and Eq. (14) for tube flow is restricted to the case where transverse (for slit) and radial (for tube) component of the pressure drop is negligible in comparison to the axial pressure drop in an infinitesimal section dz along the channel length. Further, the fractional recovery in this infinitesimal section should be sufficiently less than unity. This is usually the case in membrane cross-flow filtration.

Fig. 1 shows the predictions of the pressure drop as a function of non-dimensional distance along the slit (z/h) through a slit of height 2 mm, $W = 1$ m, $N_{Re} = 500$ and water as the fluid. The wall velocity corresponds to a membrane with a flux of 100 lmh at 3 bar pressure, or in other words, a membrane permeability, A of 9.17×10^{-11} m/s Pa. As can be seen, the predictions for constant wall velocity (Eq. (4) in this work) agree very well with Berman's solution [1]. For constant wall permeability (Eq. (10) in this work), there is increased pressure drop compared to the constant wall velocity since the total permeate produced over a length z reduces because of a drop in the local fluid pressure as it flows through the slit.

In Fig. 1, $z/h \sim 4500$ corresponds to about 100% recovery for the case of constant wall velocity while the corresponding recovery for constant wall permeability is around 65%.

In order to benchmark the pressure drop prediction using Eq. (10), computational fluid dynamics (CFD) calculations were carried out for the case of constant wall permeability for the slit geometry corresponding to Fig. 1. The CFD routine was implemented using

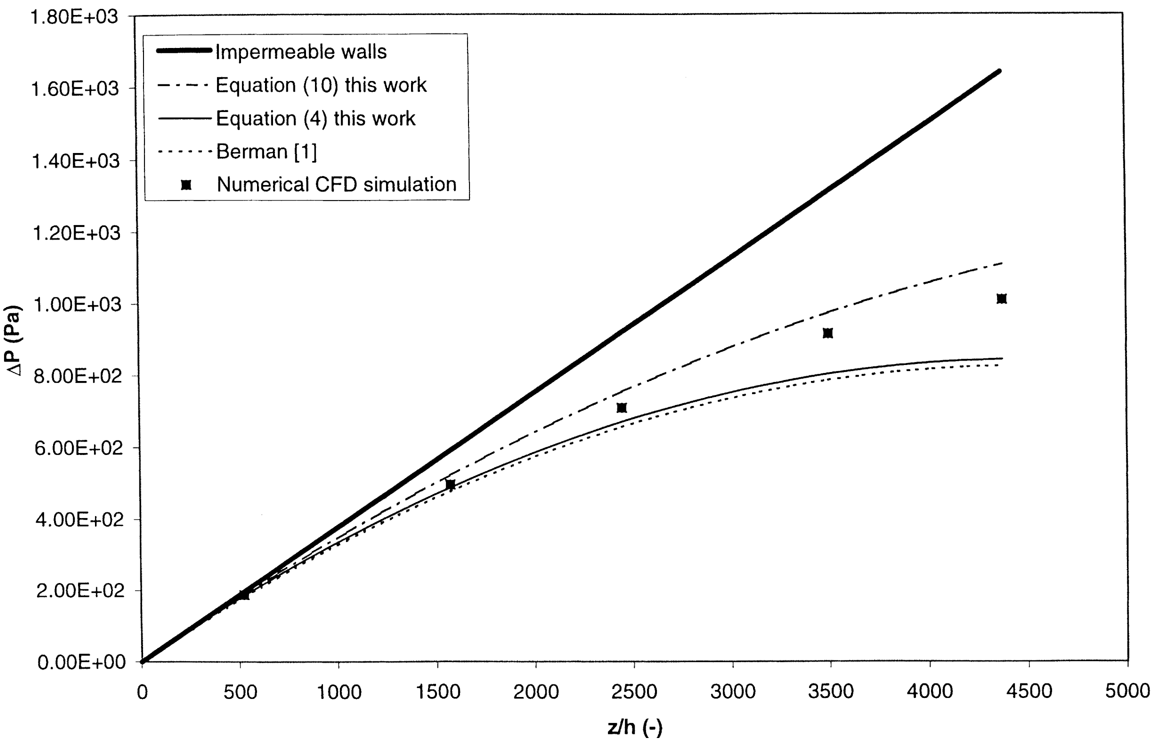

Fig. 1. Pressure drop in a rectangular channel with impermeable walls, constant wall velocity, constant wall permeability and numerical CFD simulation for constant wall permeability (model parameters in the text).

a commercially available package, PHOENICS. The accuracy of the numerical routine (sufficient grid density, etc.) was verified by comparing numerical CFD predictions for constant wall *velocity* with Berman's [1] published solution. The grid density was increased till the CFD solution was within 99% of Berman's result.

As can be seen from Fig. 1, pressure drop prediction using Eq. (10) for constant wall permeability is within 2–10% of the numerical CFD solution (shown by filled squares in Fig. 1). The agreement is very good for low values of z/h corresponding to low recoveries. At higher values of z/h (recovery $\sim$ 60%), Eq. (10) over predicts the pressure drop by about 10%.

In almost all of the models for fluid flow across semi-permeable membranes (slit or tube geometry), constant membrane permeability is used to model the wall velocity [4–9]. Eqs. (10) and (14) from this work can be used to benchmark numerical routines with constant wall permeability for slit and tube geometry, respectively, and for quick engineering estimates of pressure drop through membrane systems. Such expressions are more appropriate for membrane transport than the constant wall velocity expression derived by Berman [1]. Berman's solution would be valid in the reverse-osmosis range where the permeate velocity is essentially constant along the channel length. For ultrafiltration (UF) and microfiltration (MF), however, the channel pressure drop would be a significant fraction of the inlet pressure and the expressions presented in this work for constant wall permeability would be more accurate.

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