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AN INTRODUCTION TO ANALYTICAL STRIP TRIANGULATION, WITH A FORTRAN PROGRAM

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PHOTOGRAMMETRIC RESEARCH

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Acoustics

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Interferometry

Mechanics

Photogrammetry

Radiation Optics

X-Rays and Nuclear Radiations.

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Introduction

In photogrammetric mapping, an extensive use is made of procedures which serve to increase the number of ground-control points established by field surveying.

For a long time, the most accurate of the available procedures consisted in aerial triangulation of strips of photographs in first-order plotting instruments followed by transformation of the obtained strip coordinates to the required geodetic or map-coordinate system.

Analytical aerial triangulation is an alternative to the triangulation in an instrument. It consists in computing the map coordinates of terrain points directly from measurements of the coordinates of their images in the planes of aerial photographs. The orientation of each photograph of a strip at its moment of exposure with respect to the others is computed in a rectangular three-dimensional coordinate system. With this orientation, rays from corresponding images of terrain points in each two successive photographs intersect, and the coordinates of all these points of intersection are computed. Unless the computation is performed in the ground-control system and the conditions for the available ground-control points have been taken into account, the analytical triangulation is also followed by transformation to the ground-control system.

Analytical triangulation has been considered a potential method since the beginning of photogrammetric mapping. However, before the advent of the electronic computer, the required computations were too time-consuming. The electronic computer has made analytical triangulation a practical possibility.

Analytical aerial triangulation has a number of advantages over triangulation on first-order plotting instruments.

A greater accuracy can be achieved because the measured coordinates can be corrected for all determinable errors in the position of the photographic image. Film distortion can be taken into account when a grid plate is used in front of the negative or, to a lesser degree, when the camera is provided with a sufficient number of fiducial marks. Lens distortion can be compensated, limited only by the accuracy with which the camera has been calibrated and its stability. In instrumental triangulation this is not the case, or at least not to this extent: for example, no corrections can be given for irregular film distortion and for asymmetric lens distortion. Corrections can be applied also for distortion caused by atmospheric refraction. If desired, corrections can be given to eliminate curvature of a strip due to curvature of the earth.

A greater accuracy will be achieved also because analytical triangulation is not restricted by some of the limitations of instrumental triangulation. Triangulation instruments, however accurate, always have their imperfections. The bundle of rays, defined by the image points in a photograph, is not reproduced with mathematical precision. The model of the terrain obtained from a strip of photographs will therefore always be more or less distorted.

In analytical triangulation the bundles of rays are defined by mathematical formulas. The accuracy of the computations is limited only by the number of decimal places used. The only instrumental errors that occur are those in the reading of coordinates on the stereocomparator. Since a precise stereocomparator is a much simpler machine than a first-order plotting instrument, the sources of instrumental errors are fewer in number and their effect can be kept much smaller.

Another limitation of the first-order plotting instruments is in the accuracy with which relative orientation can be established. An approximate orientation is established first. It is then adjusted, either empirically or by a numerical procedure. With the empirical procedure, the result depends more or less on the preference of the operator as to the extent to which he should go in reducing the parallaxes and as to which orientation elements to use. With the numerical procedure, the corrections to the orientation elements are computed unambiguously from observed parallaxes. Application of these corrections to the instrument readings, however, does not generally bring about the expected change in the parallaxes. This is caused by the fact that corrections to the orientation elements cannot be made with mathematical precision. As a result, small but perceptible systematic parallaxes are often left. In analytical triangulation this is not the case: the relative orientation as defined by observed coordinates can be established with any required degree of accuracy.

Furthermore, in instrumental triangulation, only a limited accuracy is reached in the centering of the photograph in the plate holder. This results in a distortion of the bundle of rays. In analytical triangulation, the centering is computed from the readings of the fiducial marks and does not depend upon the positioning of the photograph in the plate holder.

Analytical triangulation also promises economical advantages. A higher accuracy will make it possible to reduce the number of ground-control points, and thus the cost of field surveys, without reducing the accuracy of the produced maps. The cost of the photogrammetric equipment can also be lower. A precise monocomparator is a much simpler instrument than a first-order plotter and is much less expensive. At the present time, the price of a precise stereocomparator is very high but eventually it will have to come down to a level which reflects the relative simplicity of this instrument.

The above advantages make analytical triangulation an attractive proposition. Accordingly, as early as 1953, a method of analytical triangulation was developed at the Photogrammetric Research Section of the Division of Applied Physics of the National Research Council of Canada.

This method was programmed for the Ferut electronic computer at the University of Toronto, one of the very few electronic computers in Canada at that time [1]. Subsequently, the method was programmed for the IBM 650 and the program was adapted for use on the IBM 1620 [2]. Card decks of this program have been supplied to the research organizations and mapping agencies who requested this.

With small modifications, the method has recently been programmed in FORTRAN. Versions of the FORTRAN program are now used on the IBM 1620 (40000 digits and floating-point hardware required) and on the IBM S/360.

The first chapter of this publication gives a short analysis of different analytical triangulation procedures with special emphasis on the procedure used in the FORTRAN program. More elaborate analyses can be found in references [3] and [4]. The following four chapters treat the mathematical formulation while the last chapter contains a description of the FORTRAN IV program, a listing of the statements, and operating instructions.

I. Discussion of procedures

1. Photograph coordinates

The process of analytical triangulation starts with the reading of the coordinates of corresponding image points on a comparator.

These readings are first converted to photograph coordinates with origin in the principal point. The photograph coordinates are corrected for the effects of film distortion, lens distortion, and refraction. Corrections for earth curvature may be given if the strip coordinates are to be directly transformed to or produced in the map coordinate system, without some geodetic system as an intermediate link.

2. Orientation procedure

From the corrected photograph coordinates, the map coordinates of the required points must be computed. This computation is most conveniently performed using a spatial rectangular coordinate system. In this system, the absolute orientation of each photograph is determined. Subsequently, the spatial coordinates of all measured points are determined by intersecting corresponding rays: If the spatial coordinate system is not identical with the map coordinate system, this computation is followed by transformation of the obtained coordinates to that system.

The absolute orientation of a photograph is expressed by six elements, as for instance the three rectangular coordinates of its projection centre and three parameters that define the position of the photograph axes with respect to the axes of the coordinate system.

These six elements must be determined from the conditions which the projecting rays through corresponding image points must fulfill.

There are three types of conditions:

i The projecting rays from corresponding image points in two consecutive photographs must intersect.

ii The rays from corresponding image points in three consecutive photographs must intersect at one point. This condition can be expressed by specifying that two pairs of corresponding rays must intersect at the same distance below the common photograph.

iii The given coordinates of a ground-control point must satisfy the equations of the rays from its image points.

The conditions of the first two types express relations which must exist between the orientation elements of adjoining photographs. They determine primarily the relative orientation of these photographs.

As a result, the absolute orientation of each photograph cannot be determined independently of that of the others. It can be determined only by one of the two following procedures: either successively for one photograph after the other or for all photographs simultaneously.

The first of these procedures resembles the triangulation in a plotting instrument. Independently for each strip that is triangulated, it produces coordinates in a three-dimensional coordinate system which is not related to the ground-control system. This strip triangulation procedure is the subject of this publication.

This procedure breaks the computation up into small steps. An arbitrary orientation of the first photograph is assumed. The orientation of each following photograph is then computed in succession. Here, the procedure used in instrumental triangulation is followed: the orientation consists in the relative orientation of each photograph with respect to the preceding one followed by scaling of the resulting model. The relative orientation can be established by making five pairs of corresponding rays intersect. The resulting model can then be scaled to the preceding one by making one height or one distance in the two models equal. Generally, more points will be measured in each model than the minimum that is necessary to establish the relative orientation and the scale. This will be done partly as a check on errors and partly to increase the accuracy. The relative orientation and the scale are then adjusted separately.

As an alternative in this procedure, the six orientation elements of a photograph could be computed simultaneously. For this computation, six independent condition equations are necessary [5, 6]. One possibility would be the measurement of five pairs of corresponding points, specifying that four of the pairs of corresponding rays need merely intersect while the fifth must intersect in the point established in the preceding model. A second possibility would be the measurement of four pairs of corresponding points, specifying that two pairs of corresponding rays need merely intersect while the other two must intersect in points established in the preceding model. If more measurements are available, an adjustment could be carried out for all six elements simultaneously using all available pairs of corresponding points and all available points from the preceding model.

This alternative has a disadvantage. Errors in the orientation of a photograph result in model deformation, especially in height. If this occurs, and two or more well separated points in such a model are used in the adjustment of the next model, that model will be deformed accordingly, causing errors in all orientation elements of its second photograph. As a result, deformation of one model causes deformation of all following models in succession. Consequently, each model is affected by deformation of all preceding models, but not by deformation of any following model. Therefore, the result of the triangulation depends on the choice of the model used to start the triangulation, that is, in practice, upon the direction of the triangulation. The disadvantage is sufficient to reject this alternative.

The second of the above procedures consists in the computation of the elements of absolute orientation of all photographs simultaneously, using all available condition equations for intersecting corresponding rays and, if one wishes, all those for ground-control points [7, 8]. This involves the simultaneous solution of as many equations as there are elements of absolute orientation, i.e., six times the number of photographs.

A redundant number of measurements should be available and a rigorous method of adjustment such as the method of least squares should be applied. In the method of least squares, the condition equations serve for the computation of normal equations. Since the condition equations express only relations between the orientation elements of adjoining photographs, the non-zero elements in the matrix of the normal equations are contained in a band along the diagonal. The required storage space and computation time are roughly proportional to the number of models. The computation and solution of the equations poses no serious problem.

If this triangulation procedure is used, the condition that three corresponding rays must intersect in one point does not give rise to the objectionable one-directional error propagation. However, the computations require a multiple of the data storage and computation time required by the strip triangulation procedure.

The first three computers used by the Photogrammetric Research Section made it necessary to use the strip triangulation procedure. Ferut, the first computer, had sufficient storage space for the simultaneous computation, but could not operate for the required length of time without breakdowns. Even using the step-by-step procedure, breakdowns occurred and parts of the triangulations had to be repeated to cross gaps. The IBM 650, which was the second computer, had at first only 2000 words. This computer, with a 4000-word drum, and the IBM 1620 with 40000 decimal digits, which was subsequently used, had sufficient storage space for the normal equations of the simultaneous solution, but not for all data. The required iterative solution and the subsequent intersecting of rays would have necessitated reading in the data several times. This would have been a somewhat cumbersome procedure and the computation time would have been rather long.

The FORTRAN IV program described in this publication is also based upon the strip triangulation procedure. Initially, a FORTRAN II version was prepared for the IBM 1620 with 40000 digits and floating-point hardware. Subsequently, the FORTRAN IV version has been prepared for the IBM S/360 which in July 1965 has replaced the IBM 1620 at the N. R. C. laboratories.

3. Iterative solution of the condition equations

Analytical formulation of the conditions of intersection produces condition equations that are non-linear in the orientation elements. To solve these equations, they must be differentiated and the resulting linear equations must be used for the determination of the orientation elements in an iterative procedure. Two procedures are possible:

i. The assumed approximations of the orientation elements which are substituted into the differential equations are the same for each iteration. Since in the case of strip triangulation the orientation elements can be chosen in such a way that they are small quantities, the value zero may be chosen as the approximation for each.

When this procedure is used, the points for relative orientation are often chosen in fixed positions in a regular pattern [9, 10]. The coefficients and the solution of the equations can then be computed in advance using desk computers.

This gives the corrections to the approximate values of the orientation elements as linear functions of the want of intersection in the measured points. These functions are used for all models of a strip. The advantage of this method is the small amount of computation required per iteration. Its disadvantages are the large number of iterations that is required if the assumed approximations of the orientation elements differ much from the correct values and the restriction that is imposed upon the position of the orientation points.

It is possible to use this procedure without imposing this restriction [22]. In that case, the electronic computer must compute and pre-solve the linear equations once for each model. Especially in the case of incomplete models and of relief, where the points cannot be chosen in a regular pattern, this will improve the convergence of the iterative procedure.

ii. Alternatively, the coefficients can be computed for each iteration using the latest approximate values of the unknowns and the actual positions of the points in the photographs. In this case, the electronic computer must compute and solve the linear equations for each iteration. Because the coefficients are valid not only for the actual positions of the points but also for the latest approximate values of the unknowns, this procedure requires the smallest number of iterations and it converges even if the assumed approximations differ very much from the correct values. If speed or storage space are a problem, an approximate orientation on five points can be performed first, followed by an adjustment using all points.

Because of these advantages, the second procedure has been used in the FORTRAN program. With differences in tilts of the photographs of less than two degrees, two iterations have proved to be sufficient. Even with a convergence of the photograph axes of 90°, the FORTRAN program requires only three iterations. In both these cases, the approximation used in the first iteration consists in the assumption of parallel axes.

4. Adjustment of the relative orientation

In the FORTRAN program, the relative orientation is based directly upon the condition of intersection of corresponding rays. This is only a matter of convenience: it involves less computation than basing it on the condition that the Y-parallax or the shortest distance between corresponding points must be equal to zero [3].

The adjustment of the relative orientation is performed with the method of least squares. It is based upon the requirement that the sum of the squares of the corrections to the photograph coordinates which make the rays intersect must be a minimum.

The formulas have been developed for the case of unequal accuracy and correlation between the photograph coordinates. In practice, equal accuracy and freedom from correlation will often be assumed either for the sake of simplicity or because no reliable values are available. The FORTRAN program contains the option of either assuming equal accuracy and freedom from correlation or using an experimental formula derived for a wide angle camera with a 6" focal length.

5. Intersection of rays

After the orientation of each photograph, the projecting rays from two corresponding points will intersect only if the corrections to the photograph coordinates which follow from the adjustment of the relative orientation are actually applied.

In the N.R.C. programs, these corrections are not computed. Instead, in the earlier programs, the procedure in instrumental triangulation has been followed: the strip is triangulated roughly in the X-direction. The X- and Z-coordinates of a point are defined as being equal to those of the points on the corresponding rays at the height where their X-parallax is equal to zero. The Y-coordinate is computed as the mean of the Y-coordinates of those points, the Y-parallax as the difference. For vertical photographs and on the assumption of equal accuracy of the coordinate readings and freedom from correlation, this procedure gives the same point as is obtained after correction of the photograph coordinates.

In the FORTRAN program, the point of intersection is defined as the point midway between the rays on their line of shortest distance. The want of correspondence is defined as their shortest distance.

It can be a matter of opinion which point is the best: the least-squares point, the Y-parallax point or the line-of-shortest-distance point. If the want of correspondence is smaller than 10μ at photograph scale, the difference is negligible, except perhaps for points in the corners of super-wide-angle photographs. If the want of correspondence is considerably larger, either the point or the whole triangulation is not of a good quality and the question of best choice can only be of academic interest. The first and the third definition have the advantage that the obtained point is independent of the orientation of the strip.

6. Transformation and adjustment

If, as with the FORTRAN program, the strip coordinates are computed with respect to a preliminary coordinate system, these coordinates must subsequently be transformed to map coordinates and terrain heights. This can be done either via coordinate systems on the earth or directly. The direct way is the simpler one. A rectangular three-dimensional coordinate system is then assumed, having as coordinates the two coordinates of the map projection system and the terrain heights.

A separate program has been written for this transformation and adjustment of triangulated strips. It is described in reference [11].

II. Photograph coordinates and corrections

1. Conversion from comparator measurements to photograph coordinates

Various stereocomparators and monocomparators are now available for performing the measurements needed in analytical triangulation.

In most of these instruments, the position of an image point in the plane of a photograph can be measured with respect to a rectangular coordinate system which has a sufficient range to cover the whole photograph. Usually, the origin of this coordinate system is outside the photograph.

The analytical triangulation requires coordinates with the origin in the principal point. These photograph coordinates are obtained by subtracting the instrument coordinates of the principal point from those of the image points.

Usually, the principal point is not marked on the photograph and, consequently, is not measured. It is then necessary to measure the fiducial marks and to compute each coordinate of the fiducial centre as the mean of the corresponding coordinates of the fiducial marks. The coordinates of the principal point are then derived from those of the fiducial centre with the help of the calibration data of the camera. For practical purposes, these two points can usually be considered to be identical.

With some stereocomparators, for the right photograph parallaxes are read instead of coordinates. Such stereocomparators can be used for the measurement of vertical aerial photographs. The parallaxes are read on short screws which can be made more accurate than a screw which covers the whole range of a photograph. The parallaxes must be converted to instrument coordinates by adding them to or subtracting them from the coordinates read simultaneously for the left photograph.

Some comparators are equipped with a large number of measuring marks in the pattern of a rectangular grid. Each image point is then measured with the nearest measuring mark. At the N.R.C. laboratories, for instance, a monocomparator has been developed which is equipped with 12 × 12 measuring marks placed at 20 mm intervals. Such comparators have measuring screws which are not much longer than the distance between adjacent marks. Instrument coordinates are here obtained by adding the measurements made with the screws to the calibrated coordinates of the used measuring mark.

At N.R.C., the computation of the instrument coordinates of the principal points from the measurements of the fiducial marks is performed with a desk calculator. The conversion from instrument coordinates to photograph coordinates is included in the triangulation program.

The measurements made with the Zeiss Jena stereocomparator at N.R.C. require the conversion from parallaxes to instrument coordinates. The N.R.C. monocomparator measurements must be corrected for deviations in the position of the measuring marks from an ideal 20 mm grid. These two computations are also performed by the computer. They require small additions to the regular program which have not been included in the listing of the FORTRAN statements.

2. Corrections for film distortion

A few photogrammetric cameras are equipped with a register glass with a grid in the focal plane. This grid is therefore reproduced on the negative. For each image point, a pointing can now be made at the point itself and at one or more of the nearest grid intersections. Subtraction of the coordinate readings for image point and intersection gives the coordinates of the image point, with a grid intersection as origin. These coordinates are added to the calibrated coordinates of the grid intersection and the resulting coordinates are treated as instrument coordinates. In this way, the effect of film distortion on these coordinates is largely eliminated.

In the absence of a register glass with grid, measurement of the fiducial marks can give an indication of the distortion. Unfortunately, most cameras have only four fiducial marks and these marks are not clearly defined. They are not sufficient for a reliable determination of film distortion over the whole area of the photograph. Still, the measurements can be used to eliminate at least part of the systematic distortion.

At N. R. C., measurement of the distances of fiducial marks is used to determine average values of the change of scale of the photographs of a strip in the direction of the photograph coordinates. These average values are used to determine correction factors which the FORTRAN program can apply to the photograph coordinates.

3. Lens distortion

The distortion of a bundle of rays by a lens causes displacement of the images of the measured points with respect to their ideal positions. The latter follow from an assumed position of an undistorted bundle in the image space. The displacement of the images is called the lens distortion. It can have radial and tangential components.

The calibration of a camera is the procedure of determining the lens distortion, the position of the principal point, and the focal length. Knowledge of these quantities makes it possible to construct an undistorted bundle in the image space.

The principal point can be defined in various ways. At N. R. C., the principal point of autocollimation is used. This is the point where a ray which in the object space is perpendicular to the plane of the photograph intersects that plane.

The centre of the perspective bundle of rays in the image space is placed on the perpendicular in the assumed principal point, and at a distance equal to the calibrated focal length f from the plane of the photograph.

Let α be the angle which a ray in the object space makes with the perpendicular and let r be the distance from the principal point to the point where the ray in the image space intersects the photograph. The radial distortion Δr is the radial distance between the actual and the ideal point of intersection, and therefore:

$$\Delta r = r - f \tan \alpha \qquad \dots \ (3.1)$$

This equation shows clearly that the radial distortion is a function of the assumed value of the focal length. At N. R. C., that value of the focal length is determined which makes the maximum difference between the values of the radial distortion and standard reference values for the lens as small as possible. For cameras for which reference values are not available, that value of the focal length is determined which makes the maximum value of the distortion as small as possible. That value of the focal length is called the calibrated focal length.

Finally, the orientation of the bundle in the image space can be further fixed by specifying that in the principal point the lens distortion is equal to zero. Using the principal point of autocollimation this means that the ray which is in the object space perpendicular to the plane of the photograph is in the image space also perpendicular to this plane and intersects it in the principal point. Equation (3. 1) can now be used to compute the radial distortion for rays of known angles α from measurements of the radial distances r.

With the recently installed camera calibrator at the N. R. C. laboratories, the radial distortion along the four half-diagonals to the corners of the photograph can be determined at angular distances of 2 13/16° and multiples of this till 59 1/16° from the principal point. Occasionally, the radial distortion will be determined also along the four half-diagonals to the middle of the sides. The tangential distortion is considerably smaller than the radial distortion and is usually not determined.

The distortion caused by a manufactured lens is in practice not the same as the theoretical distortion inherent in the lens design. This is a result of a small decentering of the lens components and of other manufacturing defects. The difference is roughly equal to the effect of adding a small prism to the lens. As a result, the principal point of autocollimation does not coincide with the point where its defining ray in the object space intersects the plane of the photograph. Also, the distortion is asymmetric with respect to the principal point of autocollimation, and tangential distortion occurs.

The distortion can be referred to a different principal point. This implies that the centre of the perspective bundle in the image space is shifted to the perpendicular in the new principal point and that the bundle is rotated by the amount which makes rays to points in the area of the principal points continue to intersect the plane of the photograph in the same points. As a result of the shift and the rotation of the bundle, the pattern of the distortion in the plane of the photograph will change.

Let the shift and the rotation be made in such a way that the asymmetries in the radial distortion become as small as possible. The new principal point is then called the principal point of best symmetry.

If a shift of the perspective centre parallel to a diagonal is called a, the associated rotation $- a/f$, and a shift perpendicular to the plane of the photograph b, as shown in Figure 1, the resulting radial shift of the point of intersection of a ray and the diagonal is

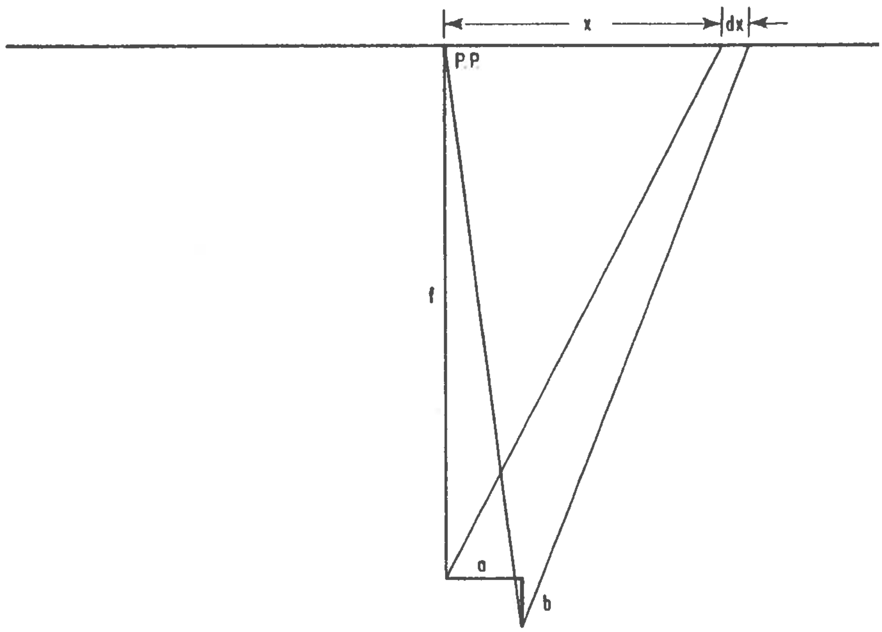

Figure 1. The effect of changes in the choice of principal point and calibrated focal length upon the radial distortion.

$$dx = a - \frac{x^2 + f^2}{f^2} a + \frac{x}{f} b \qquad \ldots (3.2)$$

Each of the three parameters contributes one term to this equation.

The radial shift dx, the distance x, and the parameters a and b each have a positive direction. The positive directions have been chosen in such a way that the values in Figure 1 are all positive. On the other hand, the radial distance r is always positive and the radial distortion is positive when it is directed away from the principal point.

One can specify that for two points on opposite sides of the principal point and at equal distances from it the shifts dx must cancel the radial distortions valid for the principal point of autocollimation. If the radial distortions at $x_1 = +r$ and at $x_2 = -r$ are called Δr_1 and Δr_2, respectively, the required radial shifts are $dx_1 = -\Delta r_1$ and $dx_2 = +\Delta r_2$. Substituted into equation (3.2), this produces two equations from which the required shifts a and b can be computed. The solution is:

$$2\frac{r^2}{f^2} a = \Delta r_1 - \Delta r_2$$

$$-2\frac{r}{f} b = \Delta r_1 + \Delta r_2 \qquad \ldots (3.3)$$

The shifts a and b can thus be computed for each pair of radial distortions. Usually they will be different for each pair and, therefore, adjusted values should be computed from the equations (3.3) by the method of least squares. This gives

$$a = \frac{\Sigma \left(2\frac{r^2}{f^2}(\Delta r_1 - \Delta r_2)\right)}{\Sigma \left(2\frac{r^2}{f^2}\right)^2} \qquad \ldots (3.4)$$

If a calibrated focal length has already been computed, the shift b is not needed.

The computation of the shift a can be performed for the two main diagonals. This gives the position of the principal point of best symmetry with respect to the previously used principal point. Table 1 gives an example of this computation applied to a 6" Hilger and Watts F.105 camera with a Wild Aviogon lens and register glass.

Table 1 An example of the effect of the choice of principal point upon the radial lens distortion

Angle	Radial distortion along four half-diagonals									
	with p.p. of autocollimation					with p.p. of best symmetry				
	NW	SW	SE	NE	mean	NW	SW	SE	NE	mean
10°	+ 2μ	+ 4μ	+ 4μ	+ 3μ	+ 3μ	+ 2μ	+ 4μ	+ 4μ	+ 3μ	+ 3μ
20°	- 1	+ 4	+ 5	+ 4	+ 3	+ 1	+ 5	+ 3	+ 3	+ 3
30°	-14	- 6	- 5	- 4	- 7	- 5	- 4	-14	- 6	- 7
40°	-12	0	+12	+15	+ 4	- 2	+ 4	+ 2	+11	+ 4
45°	+18	+17	+42	+24	+25	+32	+23	+28	+18	+25

Distance from fiducial centre to p.p. of autocollimation: less than 10μ.
Shift to p.p. of best symmetry: NW-SE 14μ, NE-SW 6μ.

It should be noted that this computation is based upon the assumption of an error-free measurement of an image produced at the principal point. If one does not wish to assume this, the rotation of the perspective bundle should not be rigidly connected with the shift a. This leads to an equation for each of the measured points in which the shift and the rotation occur as independent unknowns.

The FORTRAN program contains provision for correction of the photograph coordinates for symmetrical radial lens distortion. At N.R.C., this distortion is taken to be the mean of the radial distortions along the half-diagonals to the four corners. Usually, the fiducial centre is accepted as the principal point.

For a good camera, this procedure leaves residual radial distortions that are not larger than a few microns. This is caused by the fact that when the principal point of best symmetry is used, the actual distortion along each half-diagonal tends to differ only little from the above mean. However, where the highest possible accuracy is needed, it will be advisable to apply corrections for asymmetrical radial distortion and for tangential distortion.

Values of the radial distortion at distances from the principal point where the camera calibration does not provide them can be found by plotting the available values against the radial distance and drawing a smooth curve through the plotted points. The standard distortion curve should be used as a guide, especially at the ends of the diagonals where sufficient calibration data are often not available.

It is often difficult to estimate the best position of some parts of the curve with an accuracy of one or two microns. This causes a small amount of arbitrariness which can be avoided as follows.

Regard either the deviations of the distortion from the standard values for the lens or the distortions themselves at equal intervals along a half-diagonal as unknowns which shall be computed. Take the intervals so small that within each interval the distortion can be treated as a linear function of the position. Formulate the condition equations which specify that at the points where the camera calibration has provided values for the distortion, the distortion should have these values. Formulate also condition equations which specify that the computed distortion or its deviation from the standard values should vary in a smooth manner. This can be done, for instance, by specifying that the values of each three successive unknowns, plotted against the radial distance to the principal point, should lie on a straight line. This procedure leads to more condition equations than there are unknowns. They can be solved by the method of least squares.

4. Photogrammetric refraction

4.1 Standard atmosphere and flat earth

On their paths from the terrain to the camera, the light rays pass through air of decreasing density. As a result, they are refracted away from the vertical.

As shown in Figure 2, left, this refraction causes a small angle at the camera between the ray from a terrain point and the straight line from this point. The straight line makes a smaller angle with the vertical through the perspective centre of the camera than the ray makes at this centre. Consequently, the refraction causes a displacement of the photographic image away from the nadir point in the photograph.

This angle at the camera is called the photogrammetric refraction. It is a function of the refractive index of the air in all the points along the ray. The refractive index is a function of temperature, pressure, humidity and CO_2-content or, in short, of the density of the air.

Since these quantities cannot be measured along a whole ray, it is convenient to assume that the photogrammetric refraction in the actual atmosphere

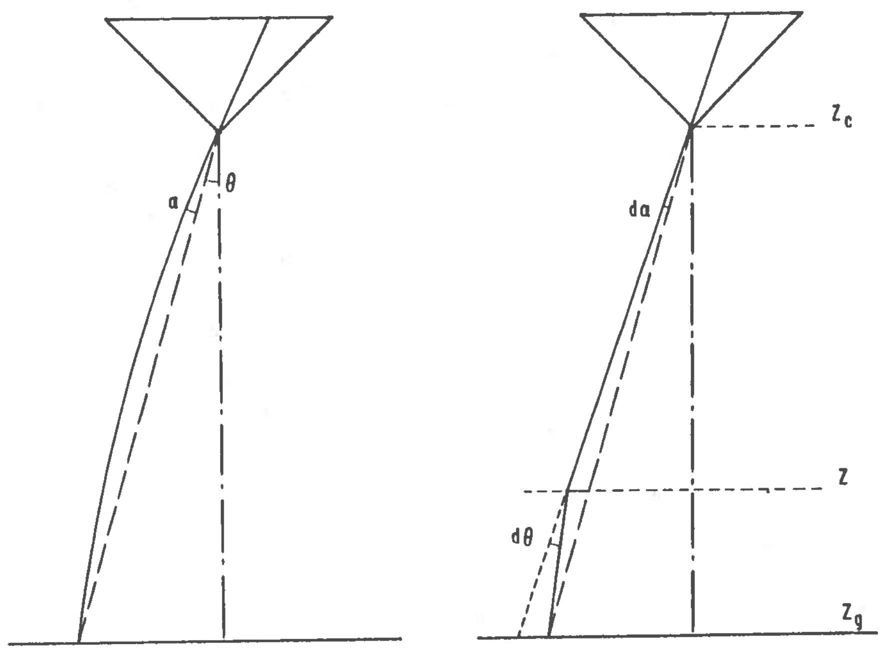

Figure 2. Refraction in the atmosphere and at a boundary between layers of different density.

is the same as that in one of the present standard atmospheres. These are:

i. The ICAO Standard Atmosphere, 1952 of the International Civil Aviation Organization [12].

ii. The ARDC Model Atmosphere, 1959 of the Air Research and Development Command of the U.S. Air Force [13].

iii. The U.S. Standard Atmosphere, 1962 of the U.S. Committee on Extension to the Standard Atmosphere [14].

Up to 20 km, these three atmospheres are practically the same. The latter two extend beyond 20 km, and up to 32 km the difference in their densities increases with increasing height to 3.5%.

Bertram [15] gives a simple method for computing the photogrammetric refraction in a standard atmosphere, using a table for the density. This method will here be followed in principle. However, the necessary formulas will be derived from Snell's well-known law of refraction instead of from the velocity of wavefronts, as Bertram does. Further, the table for the refraction will be computed in a simpler way and will give values of the refraction at all multiples of the lowest flying height instead of at odd multiples only. Finally, Bertram's formula for computing the refraction when the ground level is above sea level will be replaced because it leads to gross errors.

For the computation of the refraction, the atmosphere may be assumed to consist of a series of thin concentric shells, each of constant density. These densities decrease with increasing height of the shells. In this model of the atmosphere, refraction is caused by the changes in density at the boundaries of the shells.

According to Snell's law of refraction, for a light ray which pierces a boundary, the product $n \sin\theta$ is the same on both sides of the boundary:

$$n \sin\theta = \text{constant} \qquad \ldots (4.1)$$

Here, n is the index of refraction and θ is the angle between the light ray and the perpendicular to the boundary at the point where it is pierced by the ray.

The angle of refraction, that is the difference $d\theta$ between the values of θ on the two sides of the boundary, can be expressed as a function of the difference dn between the indices of refraction by differentiating equation (4.1). Disregarding a minus sign obtained during the differentiation, this gives the following relation between the absolute values of $d\theta$ and dn:

$$d\theta = \frac{dn}{n} \tan\theta \,, \qquad \ldots (4.2)$$

where $d\theta$ is expressed in radians.

It follows from Figure 2, right, that each refraction $d\theta$ contributes to the photogrammetric refraction the amount

$$d\alpha = \frac{Z - Z_g}{Z_c - Z_g} d\theta \qquad \ldots (4.3)$$

where Z, Z_g, and Z_c are the height of the boundary, of the ground, and of the camera, respectively. The photogrammetric refraction is the sum of the angles $d\alpha$ over all boundaries between the ground height and the height of the camera.

Since the tables for the standard atmospheres give the density but not the refractive index as a function of the altitude, it is preferable to write $d\theta$ as a function of the change in density rather than as a function of the change in refractive index.

According to textbooks on meteorological optics, the relation between density and refractive index has been determined by experiment and has been expressed in various formulas. One of the simpliest is

$$n^2 = 1 + 2c\rho \qquad \ldots (4.4)$$

where ρ is the density in, for instance, kg/m^3.

The constant c in this equation, and therefore the index of refraction, is a function of the wavelength of the light. According to Edlén [16], the formula which over the whole range of the visible spectrum agrees best with the results of experiments is for standard air:

$$(n - 1)10^7 = 643.28 + \frac{294981.0}{146 - 1/\lambda^2} + \frac{2554.0}{41 - 1/\lambda^2} \,, \qquad \ldots (4.5)$$

where the wavelength λ is measured in microns. This standard air is at 15°C, at normal pressure (760 mm Hg at 0°C), has 0.03% CO_2 (by volume at 0°C), and is dry. It has practically the same composition as the air in the standard atmospheres and the same density of 1.2250 kg/m^3 at sea level as the standard atmospheres.

If the values of the index of refraction, derived from equation (4.5) for different wavelengths, and the above value of the density are substituted into equation (4.4), the following values of the constant c are obtained.

For $\lambda = 0.42\ \mu$, $c = 0.00023004$

For $\lambda = 0.56\ \mu$, $c = 0.00022667$

For $\lambda = 0.66\ \mu$, $c = 0.00022550$

The first and the last of these values of λ are near the ends of the effective range of panchromatic film; the second one is a suitable average.

Differentiation of equation (4.4) gives

$$\frac{dn}{n} = \frac{c}{n^2}\, d\rho \qquad \ldots (4.6)$$

and, since from ground level to empty space n varies from about 1.00022 to 1, over this range and for $\lambda = 0.56$ microns

$$\frac{dn}{n} = 0.0002266\, d\rho \qquad \ldots (4.7)$$

Kaye and Laby [17] use the simplified formula $n - 1 = c\rho$. From their values of $n - 1$ and ρ, reduced to the temperature of 15°C and the wavelength of 0.56 microns, one finds $c = 0.0002261$. Leyonhufvud [18] employs a coefficient of 0.22607 for the D-line. Reduced to the same values, this gives $c = 0.0002265$.

Considering the uncertainty in the fourth significant digit of c and the limited number of digits that is needed, that digit may be left out.

Combining now equations (4.3), (4.2), and (4.7), and forming the sum over all boundaries between ground level and camera height, one obtains for the photogrammetric refraction the expression

$$\alpha = 0.000226 \frac{\tan \theta}{Z_c - Z_g} \Sigma((Z - Z_g) d\rho) \qquad \ldots (4.8)$$

In this equation, the factors which are the same for all $d\alpha$ have been placed outside the summation. The equation is identical with the one derived by Bertram.

Equation (4.8) makes the computation of the photogrammetric refraction in one of the standard atmospheres very simple. For these atmospheres, the density is listed at discrete values of the geometric height above sea level. In the above model of the atmosphere, the boundary between two shells of constant density is now chosen midway between two heights for which the density is listed and the densities of the two shells are taken to be those listed densities. In this way, each boundary produces a contribution $(Z - Z_g)d\rho$ to the sum in equation (4.8), Z being the height of the boundary and $d\rho$ being the difference between the densities at the two table heights. For each flying height for which the photogrammetric refraction is required, the contributions from all boundaries between it and ground level are added and subsequently the sum is multiplied by the factors which have been placed outside the summation.

Table 2 gives the result of this computation for a ray that makes an angle of 45° with the vertical, assuming flying heights above sea level of up to 32 km and four different ground heights. To obtain the best possible computational accuracy, densities at intervals of less than 500 meters must be used especially for the lower shells. Actually, the densities at all multiples of 100 meters from sea level to 20000 m and at all multiples of 200 meters from 20000 m to 32000 m have been used. This places the boundaries between the shells at 50 m, 150 m, 250 m, etc. The table values are accurate to within one digit of the least significant digit.

Table 2 Photogrammetric refraction in microradians for a ray at 45° with the vertical in the U. S. Standard Atmosphere, 1962

Flying height above sea level	Photogrammetric refraction for ground heights of				Flying height above sea level	Photogrammetric refraction for ground heights of			
	0.0 km	1.0 km	2.0 km	4.0 km		0.0 km	1.0 km	2.0 km	4.0 km
0.5 km	6.5				13.5 km	91.3	82.0	73.2	57.0
1.0	12.6	0.0			14.0	92.2	83.0	74.2	58.2
1.5	18.5	6.0			14.5	92.8	83.7	75.1	59.2
2.0	24.1	11.7	0.0		15.0	93.3	84.2	75.7	60.1
2.5	29.3	17.1	5.6		15.5	93.5	84.6	76.2	60.7
3.0	34.3	22.3	10.9		16.0	93.6	84.8	76.5	61.2
3.5	39.0	27.1	15.9		16.5	93.6	84.9	76.6	61.5
4.0	43.5	31.7	20.6	0.0	17.0	93.4	84.8	76.6	61.7
4.5	47.7	36.1	25.1	4.7	17.5	93.2	84.6	76.6	61.8
5.0	51.6	40.2	29.3	9.2	18.0	92.8	84.3	76.4	61.8
5.5	55.3	44.0	33.3	13.5	18.5	92.3	84.0	76.1	61.7
6.0	58.8	47.6	37.0	17.5	19.0	91.8	83.5	75.7	61.5
6.5	62.1	51.0	40.6	21.3	19.5	91.2	83.0	75.3	61.3
7.0	65.1	54.2	43.9	24.8	20.0	90.5	82.4	74.8	61.0
7.5	67.9	57.2	47.0	28.2	21.0	89.1	81.2	73.8	60.3
8.0	70.6	59.9	49.8	31.3	22.0	87.5	79.8	72.6	59.4
8.5	73.0	62.5	52.5	34.2	23.0	85.8	78.3	71.2	58.4
9.0	75.2	64.9	55.0	37.0	24.0	84.0	76.7	69.8	57.2
9.5	77.3	67.1	57.4	39.5	25.0	82.2	75.0	68.2	56.0
10.0	79.2	69.1	59.5	41.9	26.0	80.3	73.4	66.7	54.8
10.5	80.9	70.9	61.5	44.1	27.0	78.4	71.6	65.1	53.5
11.0	82.5	72.6	63.3	46.1	28.0	76.6	69.8	63.6	52.2
11.5	85.0	75.2	66.0	49.0	29.0	74.7	68.2	62.0	50.9
12.0	87.1	77.4	68.3	51.5	30.0	72.9	66.5	60.5	49.6
12.5	88.8	79.3	70.2	53.7	31.0	71.1	64.8	59.0	48.4
13.0	90.2	80.8	71.8	55.5	32.0	69.4	63.2	57.5	47.1

Table 2, cont'd. Photogrammetric refraction in the tentative region of the U.S. Standard Atmosphere, 1962

Flying height above sea level	Photogrammetric refraction for ground heights of 0.0 km	2.0 km	Flying height above sea level	Photogrammetric refraction for ground heights of 0.0 km	2.0 km
32 km	69.4	57.5	62 km	37.7	30.6
34	66.1	54.7	64	36.5	29.6
36	63.1	52.1	66	35.4	28.7
38	60.1	49.6	68	34.4	27.8
40	57.4	47.3	70	33.4	27.0
42	54.9	45.1	72	32.5	26.2
44	52.6	43.1	74	31.6	25.5
46	50.4	41.3	76	30.8	24.8
48	48.4	39.6	78	30.0	24.2
50	46.5	38.0	80	29.3	23.6
52	44.8	36.5	82	28.5	23.0
54	43.2	35.2	84	27.9	22.4
56	41.7	33.9	86	27.2	21.9
58	40.3	32.7	88	26.6	21.4
60	38.9	31.6	90	26.0	20.9
			Z > 90	$\frac{2340.5}{Z}$	$\frac{1837.4}{Z - 2}$

Values of the refraction for unlisted flying heights can be computed with sufficient accuracy by linear interpolation in the column of the required ground height. Values for unlisted ground heights can be computed by interpolation between values for listed ground heights. For accurate values, second differences must be used in the interpolation and around the flying height of 11000 m, where a discontinuity occurs, the interpolation must be made between values of the same flying height above sea level, not between values of the same flying height above ground.

It follows from equation (4.8) that for angles θ other than 45° the values of the refraction can be obtained by multiplying the table values by tan θ.

The table gives the values for the U.S. Standard Atmosphere, 1962. The values for the ARDC Model Atmosphere, 1959 are the same, except for flying heights from 21 to 25 km where they are 0.1 μrad smaller and for flying heights from 27 to 32 km where they are 0.1 μrad larger.

In the preceding, the table values for ground heights above sea level have been computed in the same way as those for the ground height at sea level. However, it is possible to compute them directly from the latter. For this purpose, the sum in equation (4.8) is written

$$\sum_{s}^{c} Z d\rho - \sum_{s}^{g} Z d\rho - Z_g \sum_{g}^{c} d\rho \, .$$

These three summations are performed over the boundaries between sea level and flying height, between sea level and ground level, and between ground level and flying height, respectively.

Substituted in equation (4.8), this gives

$$\alpha = \frac{Z_c}{Z_c - Z_g}\alpha_c - \frac{Z_g}{Z_c - Z_g}\alpha_g - .000226 \tan\theta \frac{Z_g}{Z_c - Z_g}\Delta\rho \qquad \ldots (4.9)$$

where α_c and α_g are the photogrammetric refraction at the actual flying height and at the actual ground height, both with respect to a ground height at sea level, and $\Delta\rho$ is the difference between the densities at the actual flying height and ground height.

4.ii Contribution of earth curvature

A complication which has not been considered yet is the fact that, due to the earth curvature, the verticals in any two points of a tilted light ray are not parallel.

Since the angle θ is defined as the angle which the ray makes with the vertical, it follows that along a ray this angle varies not only because of the small atmospheric refraction but also because of the much larger change in the direction of the vertical.

As a result, $\tan\theta$ in equation (4.8) is not a constant, but is different for each term in the summation. If θ_c is the angle θ at the camera and β is the angle between the verticals at the camera and at an arbitrary point on the ray,

$$\theta = \theta_c - d\theta + \beta$$

In first approximation, and neglecting $d\theta$,

$$\tan\theta = \tan\theta_c + \sec^2\theta_c\beta$$

or, again approximately,

$$\tan\theta = \tan\theta_c\,(1 + \sec^2\theta_c\,\frac{Z_c - Z}{R}), \qquad \ldots (4.10)$$

where R is the radius of the earth.

This formula can be derived also by differentiating the formula for the refraction in the atmosphere

$$n\,R\,\sin\theta = \text{constant}.$$

It follows from equation (4.10) that $\tan\theta$ in equation (4.8) should be replaced by the expression in equation (4.10), and the second factor in this expression should be brought under the summation. The same result is obtained by replacing $\tan\theta$ by $\tan\theta_c$ and adding

$$\Sigma\,(\sec^2\theta_c\,\frac{Z_c - Z}{R}\,(Z - Z_g)d\rho)$$

to the sum of products.

The computation of the sum for all flying heights and for all ground heights can be simplified by writing it as

$$\sec^2\theta_c \frac{Z_c - Z_g}{R} \Sigma ((Z - Z_g)d\rho) - \Sigma (\sec^2\theta_c \frac{Z - Z_g}{R} (Z - Z_g)d\rho),$$

in which only the simple first term is a function of the flying height.

The resulting corrections to the photogrammetric refraction for an angle $\theta = 45°$ are listed in Table 3 as a function of flying height and ground height. The corrections for other angles can be obtained from the table values by multiplication by $\frac{1}{2} \sec^2\theta$.

Table 3 Contribution of earth curvature to refraction for a ray at 45° with the vertical, in microradians

Flying height above sea level	Contribution to the refraction for ground heights of			
	0.0 km	1.0 km	2.0 km	4.0 km
5.0 km	0.03	0.02	0.01	0.00
10.0	0.10	0.07	0.06	0.03
15.0	0.18	0.15	0.12	0.08
20.0	0.26	0.23	0.19	0.14
25.0	0.33	0.29	0.25	0.18
30.0	0.39	0.34	0.30	0.22

The photogrammetric refraction, including the effect of earth curvature, is therefore derived from the table value c_1 found in Table 1 and the table value c_2 found in Table 2 by means of the equation

$$\alpha = \tan\theta \left(c_1 + \frac{1}{2} \sec^2\theta \, c_2\right) \qquad \ldots (4.11)$$

The value c_2 is so small that its contribution can practically always be neglected.

4.iii Refraction in the actual atmosphere

The last problem that has to be dealt with is the difference between the actual atmosphere and the standard atmospheres.

The temperature, pressure, and composition of the actual atmosphere are never known completely. Even if they were, the computation of the photogrammetric refraction would be too complicated. Therefore, usually the difference between actual atmosphere and standard atmosphere is neglected.

An estimate of the photogrammetric refraction based on measurements of temperature, pressure, and relative humidity at ground level can be computed as follows.

The standard atmospheres have a temperature of 15°C (59°F) and a pressure of 760 mm Hg at sea level and they are dry. According to the law of Boyle for a perfect gas, an increase in the absolute temperature of 1 3/4%

throughout the atmosphere, that is 5°C (9°F) at sea level, causes a uniform decrease of about 1 3/4% in the density. Therefore, it causes the same decrease in the density differences in equation (4. 8) and in the photogrammetric refraction. An increase of 1 1/3% in the pressure, that is 10 mm Hg at sea level, causes an increase of 1 1/3% in the density and in the photogrammetric refraction.

The density of damp air can be found by multiplying the density of dry air by the factor (P - 0.378p)/P, where P is the pressure of the dry air and p is the pressure of the water vapour. With this formula, and a table of the saturation pressure of water vapour as a function of the temperature, the decrease in density can be computed as a function of the relative humidity. If the lower layers of the actual atmosphere have 100% relative humidity, while the pressure and temperature are the same as in the standard atmospheres, the decrease in density is 2/3% at sea level and 1/3% at 2000 m. Due to this variation in the decrease, the density differences and the photogrammetric refraction below 2000 m decrease by about 2%. If also the absolute temperatures are 3.5% higher than in the standard atmospheres, it is then 25°C (77°F) at sea level and the decrease in density is 4.5% at sea level and 4.0% at 2000 m. As a result, the density differences and the photogrammetric refraction decrease by about 7%.

5. Earth curvature

When the strip coordinates obtained by analytical strip triangulation are directly transformed to a three-dimensional rectangular coordinate system constructed from map coordinates (as easting and northing) and heights, one obstacle is met. This consists in the fact that the model of the earth presented by this coordinate system is deformed: on the earth the height of a point is the shortest distance from the point to the curved equipotential surface at sea level and in this coordinate system it is the shortest distance to the plane which contains the horizontal axes.

The deviation of the equipotential surface from a plane is considerable, even in the area of one strip. A 230 by 230 mm photograph taken with a 152.4 mm (6'') lens at a height of 6 km covers an area of 9 by 9 km. Points on this surface in the middle of the sides of this area are already 1.6 m below the plane which is tangent to the surface in the centre. Points in the corners are 3.2 m below this plane. If the photograph is the first one of a strip that is 100 km long, points at the end of the strip are 800 m below the plane.

It follows that even for a single photograph this deviation cannot be neglected. The direct transformation produces correct map coordinates and heights only if it is preceded by or accompanied by a deformation of the triangulated strip that is identical with the deformation in that system.

In the area of a strip the equipotential surface may be approximated by the sphere which has as its radius the mean radius of the spheroid in this area. This sphere must be transformed into a plane. A satisfactory transformation is obtained by changing the great circle along the axis of the strip into a straight line of true length and by changing great circles at right angles to the first into straight lines of true length at right angles to the first. This can be done in two steps: first the sphere is changed into the cylinder which is tangent to the sphere in the great circle along the axis of the strip and then the cylinder is rolled out onto a plane which is tangent to the sphere in a point of that great circle.

This deformation can be obtained directly by giving appropriate corrections to the photograph coordinates. To derive these corrections for each photograph, the above plane is chosen tanget to the surface in the nadir point of the photograph.

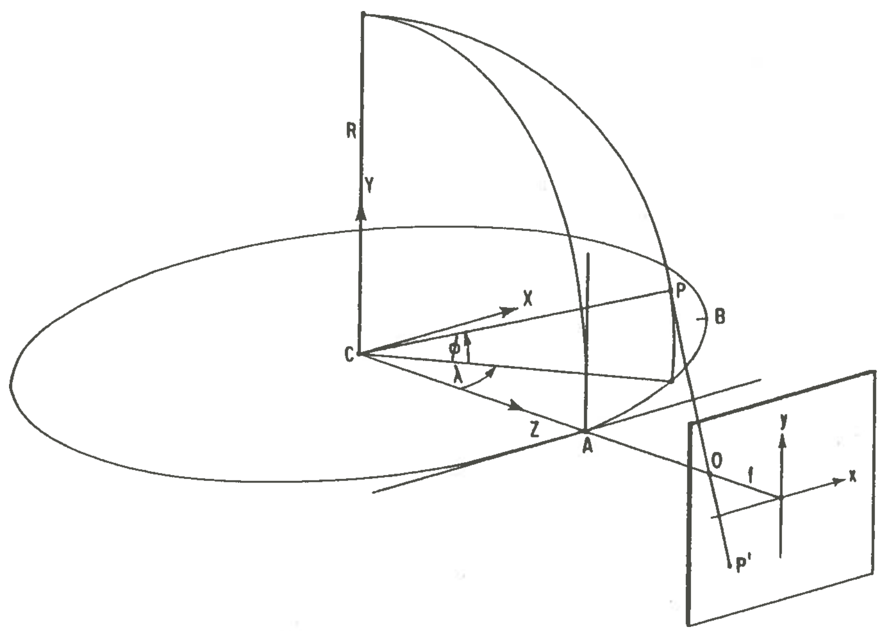

Figure 3. Elimination of the effect of earth curvature by projection of a sphere upon a tangent plane.

Figure 3 shows this situation. AB is the great circle along the axis of the strip and A is the nadir point of the photograph. A rectangular coordinate system is assumed with the origin in the centre of the sphere, the Z-axis through A, and the X-axis in the plane of the great circle.

If the great circle is the equator in a system of geographic coordinates ϕ and λ as shown in Figure 3 and the radius is called R, the geocentric coordinates of a point P on the sphere are:

$$X = R \cos\phi \sin\lambda$$
$$Y = R \sin\phi$$
$$Z = R \cos\phi \cos\lambda \qquad \ldots (5.1)$$

After the transformation of the sphere the point P is situated in the plane. Its coordinates are here

$$X_p = R\lambda$$
$$Y_p = R\phi$$
$$Z_p = R \ , \qquad \ldots (5.2)$$

in which ϕ and λ are expressed in radians.

The corrections to the geocentric coordinates which transform the sphere into the plane are thus

$$dX = X_p - X = X \left(\frac{\lambda}{\cos\phi \ \sin\lambda} - 1\right)$$
$$dY = Y_p - Y = Y \left(\frac{\phi}{\sin\phi} - 1\right)$$
$$dZ = Z_p - Z = R \left(1 - \cos\phi \ \cos\lambda\right) \qquad \ldots (5.3)$$

For the above-mentioned photograph, the maximum value of ϕ and of λ is about 150''. This makes the maximum values of dX and dY less than one millimeter. These corrections are negligible. This means that for each photograph the sphere may be projected orthogonally on the plane which is tangent to it in the nadir point of the photograph.

Therefore, in the geocentric system only the Z-coordinate needs a correction. With a negligible approximation,

$$dZ = \frac{1}{2} R \left(\phi^2 + \lambda^2\right) = \frac{X^2 + Y^2}{2R} \qquad \ldots (5.4)$$

This correction has been derived for points on the sphere. However, because before the triangulation the heights are not known, points with a different elevation will be given the same correction. As a result, a line which is perpendicular to the sphere will be shifted only. Consequently, it will not become orthogonal to the plane, as should be the case.

In the corners of the area covered by the above-mentioned photograph, the resulting errors are smaller than 0.1 m for every 100 m of difference in terrain height. They cause no y-parallaxes in a pair of photographs and, therefore, no errors in the relative orientation. They do cause x-parallaxes and, as a result, a small exaggeration of the vertical scale.

The errors could be corrected by taking heights, computed after the last-but-one iteration of the relative orientation, into account. This will be worthwhile if the corrections for asymmetric lens distortion are also taken into consideration.

6. Corrections for lens distortion, refraction, and earth curvature

Corrections to the photograph coordinates for symmetrical radial lens distortion are computed with the formulas

$$dx = x \frac{dr}{r} \text{ and } dy = y \frac{dr}{r} \qquad \ldots (6.1)$$

where dr is the radial correction.

Corrections for refraction and earth curvature are computed with the same formulas after conversion of the photogrammetric refraction α and the earth curvature correction dZ to radial corrections.

The radial correction for refraction is

$$dr = - \frac{f}{\cos^2 \theta} \alpha \qquad \ldots (6.2)$$

Since $\alpha = c_1 \tan \theta$, it follows that

$$\frac{dr}{r} = - (1 + \frac{r^2}{f^2}) c_1 \qquad \ldots (6.3)$$

Here, c_1 is the photogrammetric refraction in radians at $\theta = 45°$, taken from Table 2, and f is the focal length.

The radial correction for earth curvature can be computed with the help of Figure 4 which shows a vertical cross section through the sphere. It follows from similar triangles that

$$(\frac{H}{f} dr)/dZ = r/f. \qquad \ldots (6.4)$$

Replacing dZ by the expression in the last part of equation (5.4), this gives

$$\frac{dr}{r} = \frac{H}{2R} \frac{r^2}{f^2} \qquad \ldots (6.5)$$

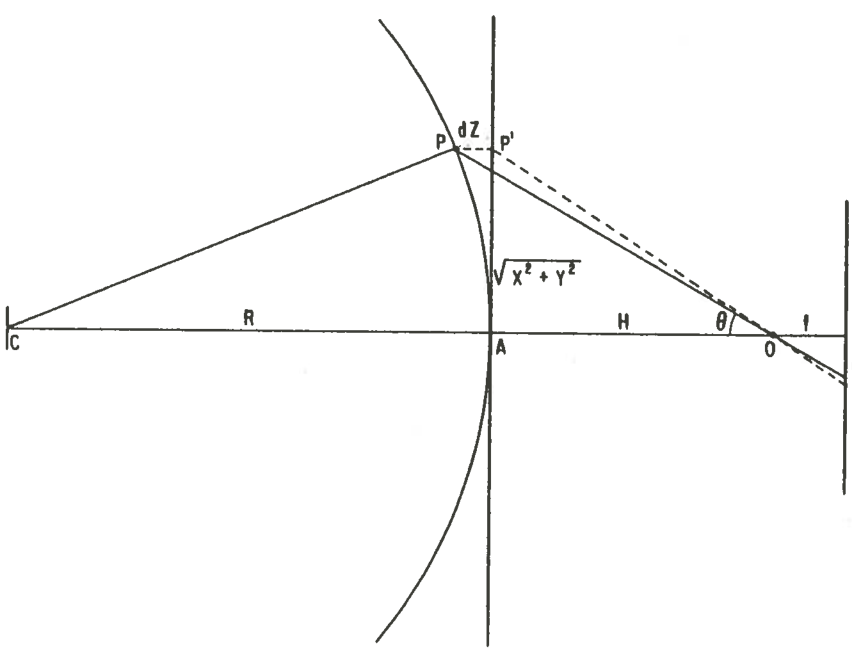

Figure 4. The radial correction for earth curvature.

The FORTRAN program derives the radial corrections for lens distortion from a table in which the value of dr is listed at a number of values of r. For each point, it computes dr by linear interpolation between the two nearest table values. The radial corrections for refraction are computed only if the value of c_I is punched in the first card for a strip triangulation, and the radial corrections for earth curvature are computed only if the flying height is punched in that card. Finally, the equations (6.1) are used to compute the corrections dx and dy from the sum of the dr/r.

The above corrections for refraction and earth curvature are valid for exactly vertical photographs. If the photographs are tilted, the corrections can be derived in the following way.

The angular correction α_1 for photogrammetric refraction is, according to equation (4.11), $\alpha_1 = c_1 \tan\theta$; the angular correction α_2 for earth curvature can be computed with the help of Figure 4 and equation (5.4) and is

$$\alpha_2 = dZ \sin\theta / \sqrt{H^2 + X^2 + Y^2} = \frac{H}{2R} \frac{\sin^3\theta}{\cos\theta} \qquad \ldots (6.6)$$

If the tilt of a photograph is τ, a ray in the direction of this tilt makes an angle $\theta - \tau$ with the camera axis. Therefore, its radial corrections for refraction and earth curvature can be computed with equation (6.2) if α is replaced by α_1 and by $-\alpha_2$, respectively, and θ is replaced by $\theta - \tau$.

As an example, the radial corrections in the direction of greatest tilt have been computed for the above-mentioned wide-angle photograph and for a super-wide angle photograph, assuming tilts of 0° and 2°. The resulting values are listed in Table 4. The change in the radial corrections is approximately proportional to the tilt.

Table 4 Effect of tilt on radial corrections for refraction and for earth curvature

Angular distance from camera axis	Radial corrections for refraction, tilt 0°	Radial corrections for refraction, tilt 2°	Radial corrections for earth curvature, tilt 0°	Radial corrections for earth curvature, tilt 2°
	f = 152.4 mm, H = 6000 m			
9°	- 1.5 μ	- 1.8 μ	0.3 μ	0.5 μ
18°	- 3.2	- 3.6	2.5	3.4
27°	- 5.7	- 6.2	9.5	11.7
36°	- 9.9	-10.7	27.5	32.5
45°	-17.9	-19.2	71.7	82.3
	f = 88.2 mm, H = 6000 m			
45°	-10.4	-11.1	41.5	47.6
59°	-32.5	-35.2	191.	216.

The tilt of a "vertical" photograph is seldom more than 2°. If the corrections are based upon the assumption of exactly vertical photographs, the error in the correction for refraction is thus not greater than one micron in the corners of the wide-angle photograph and less everywhere else. Considering the uncertainty in the refraction itself, this error may be neglected.

If the tilt is 2°, the corresponding errors in the correction for earth curvature in the corners of the wide-angle photograph are up to 10 microns. However, if two consecutive photographs have the same tilt the effect on the model is small. This may be seen by rotating photographs and model until the camera axes are vertical. Errors in the model are now caused only by the difference between the height H used in formula (6.5) and the heights of the projection centres above terrain points in the rotated model.

This shows that the absolute tilt of the photographs is of relatively little importance. The difference in tilt of successive photographs however must be small: less than half a degree to make the error in the correction for earth curvature less than 2 microns. Only then can formula (6.5) be applied safely to tilted photographs.

Consequently, if the accuracy requirements make it necessary to apply corrections for asymmetric lens distortion, the earth curvature corrections should take into account not only the height differences, as stated before, but also the tilt of the photographs in the strip coordinate system.

In the case of the above super-wide-angle photograph, the maximum errors in the correction for earth curvature are 25 microns. Therefore, here it seems to be better not to apply a symmetrical radial correction for earth curvature to the photograph coordinates. Either an asymmetrical correction should be used or the strip coordinates should be corrected instead.

III. The orientation of a photograph

1. The orthogonal transformation matrix

For each photograph of a strip, a three-dimensional rectangular coordinate system x, y, z will be assumed with the origin in the projection centre of the photograph. The x- and y-axes will be parallel to the plane of the photograph and, therefore, the x- and y-coordinates of an image point will be identical with the photograph coordinates. The z-coordinate of an image point will be equal to +f (the calibrated focal length) if the photograph is in negative position (above the projection centre) and equal to -f if the photograph is in positive position.

The strip triangulation will be performed with respect to a three-dimensional rectangular coordinate system. In this system, the orientation of each photograph can be defined by means of the three coordinates of the projection centre and of parameters which determine its attitude.

Further, for each photograph a coordinate system X, Y, Z is needed with the origin in the projection centre of the photograph and with axes parallel to the axes of the strip coordinate system.

The relation between the coordinates X, Y, Z and the coordinates x, y, z of any point in a photograph is given by the matrix equation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or, simply,

$$\mathbf{X} = \mathbf{A}\,\mathbf{x} \qquad \dots (1.1)$$

This is immediately evident when in Figure 5 the components of the position vector of a point with respect to the x, y, z coordinate system are projected upon the X-axis, the Y-axis, and the Z-axis. The elements of the first, second and third column of the matrix $\mathbf{A}$ are then seen to be the direction cosines of the x-, y-, and z-axes with respect to the X, Y, Z coordinate system.

Equation (1.1) will be used to define the attitude of a photograph. Therefore, either the nine direction cosines can be selected as the parameters which define its attitude or the direction cosines must be defined as functions of other suitable parameters.

The matrix $\mathbf{A}$ has the property that

$$\mathbf{A}^T \mathbf{A} = \mathbf{I} \qquad \dots (1.2)$$

The superscript T indicates the transpose of the matrix to which it is attached, and the matrix $\mathbf{I}$ is the unit matrix.

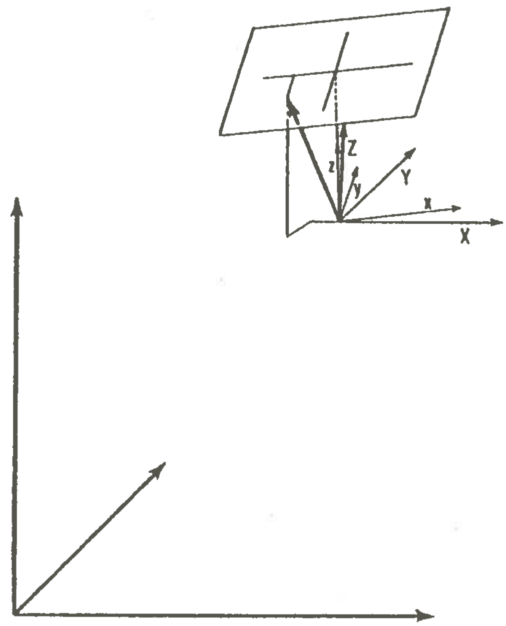

Figure 5. Coordinate systems used for the orientation of a photograph, and the position vector of an image point with its components.

This can be proved as follows. The position vectors $\mathbf{X}$ and $\mathbf{x}$ are identical, the only difference being the coordinate system in which their components are defined. Therefore, they have the same length, and consequently the sum of the products of their components is the same. In matrix notation:

$$\mathbf{X}^T \mathbf{X} = \mathbf{x}^T \mathbf{x},$$

or

$$\mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{x},$$

and

$$\mathbf{x}^T (\mathbf{A}^T \mathbf{A} - \mathbf{I}) \mathbf{x} = 0,$$

for any vector x. The first part of this equation is a polynomial of the second degree in the components of $\mathbf{x}$. It is equal to zero for any vector $\mathbf{x}$ if and only if all coefficients in this polynomial are equal to zero, that is, if $\mathbf{A}^T \mathbf{A} - \mathbf{I} = \mathbf{O}$!

Equation (1.2) states that the sum of the squares of the elements of each column of $\mathbf{A}$ is equal to 1 and that the sum of the products of corresponding elements in different columns is equal to zero. A matrix which satisfies these conditions is called orthogonal. Thus, the equation contains six independent relations between the nine elements of $\mathbf{A}$. Therefore, the elements contain only three independent parameters.

It follows from equation (1.2) that

$$\mathbf{A}^{-1} = \mathbf{A}^{T} \qquad \ldots (1.3)$$

and therefore also that $\mathbf{A}\,\mathbf{A}^{T} = \mathbf{I}$. The latter equation expresses similar relations which exist between the elements of the rows of $\mathbf{A}$.

These relations can be expressed by the following theorems:

I The sum of the squares of the elements in each column and in each row is equal to 1:

$$a_{1i}^2 + a_{2i}^2 + a_{3i}^2 = 1$$

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1 \qquad (i = 1, 2, 3)$$

II The sum of the products of the elements in corresponding positions in each two columns and in each two rows is equal to zero:

$$a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} = 0$$

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3;\ i \neq j)$$

From these two theorems, two other theorems can be derived:

III Each element is equal to its cofactor:

$$a_{il} = a_{jm}a_{kn} - a_{jn}a_{km}$$

Here, the row indices i, j, and k represent three sequential numbers of the sequence 1, 2, 3, 1, 2 and the column indices l, m, and n independently represent three sequential numbers of this sequence.

IV The determinant of the matrix is equal to +1.

The last two theorems are valid only if the two coordinate systems are either both right-handed, as in the present case, or both left-handed. The matrix is then called proper orthogonal. If one of the coordinate systems is right-handed and the other is left-handed, each element is equal in magnitude to its cofactor but of opposite sign and the determinant is equal to -1.

The transformation (1.1) has here been introduced as representing the change in the components of a position vector under a change in the choice of coordinate system. However, by drawing a position vector which has components x, y, and z with respect to the X, Y, Z system, it becomes evident that the transformation also represents the rotation of the position vector from this position to the one in Figure 5. Therefore, it can represent the rotation of the photograph about the projection centre. The coordinate system X, Y, Z is then the only one and does not change its position.

Let now two such rotations be applied in succession: $\mathbf{X}_1 = \mathbf{A}_1\mathbf{x}$ and $\mathbf{X}_2 = \mathbf{A}_2\mathbf{X}_1$. It follows from these equations that the final position vector of any image point can be found directly from $\mathbf{X}_2 = \mathbf{A}_2 \mathbf{A}_1 \mathbf{x}$ Since the two rotations do not change the length of any vector, the matrix $\mathbf{A} = \mathbf{A}_2 \mathbf{A}_1$ which represents the one-step rotation from $\mathbf{x}$ to $\mathbf{X}_2$ is also orthogonal. Consequently, the product of two orthogonal matrices is again an orthogonal matrix.

While thus to any attitude of the photograph corresponds one proper orthogonal matrix $\mathbf{A}$, it is also true that to every proper orthogonal matrix of the third order corresponds one attitude of the camera. For, since the sum of the squares of the elements in each column is equal to 1, these elements are the direction cosines of axes x, y, and z. Since the sum of products of corresponding elements in each two different columns is equal to zero, these axes are mutually orthogonal. Since the determinant of the matrix is equal to +1, this x, y, z system and the X, Y, Z system are either both right-handed or both left-handed. Therefore, this x, y, z system and the x, y, z system in equation (1. 1) are identical.

Consequently, there is a one-to-one correspondence between the possible attitudes of a photograph and the proper orthogonal matrices of the third order. Any proper orthogonal matrix of the third order constructed in any way from three independent parameters can serve as the matrix for equation (1.1).

The constructions which are most important in analytical triangulation are described in the following. A much more complete account can be found in reference [19].

2. Rotations about three mutually orthogonal axes

The matrices

$$\mathbf{R}_\omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \quad \mathbf{R}_\phi = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \text{ and } \mathbf{R}_\kappa = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (2.1)$$

satisfy theorems I and II and their determinants are equal to +1. Therefore, they are proper orthogonal.

Used as the matrix of equation (1. 1), each leaves one coordinate unchanged and therefore can move any point only in a plane which is orthogonal to the axis of the unchanged coordinate. Also, it leaves the distance from the point to this axis unchanged.

Consequently, $\mathbf{R}_\omega$, $\mathbf{R}_\phi$, and $\mathbf{R}_\kappa$ represent rotations about the X-, Y-, and Z-axis, respectively. The signs of the elements are such that a positive rotation of three-dimensional space with respect to the coordinate system means a clockwise rotation when viewing in the positive direction of its axis.

Any matrix which is the product of three matrices $\mathbf{R}_\omega$, $\mathbf{R}_\phi$, and $\mathbf{R}_\kappa$ is, according to the preceding, also orthogonal and it contains the required number of three evidently independent parameters.

If first a rotation κ, then a rotation ϕ, and finally a rotation ω is applied, the resulting attitude of the photograph is represented by the matrix

$$\mathbf{A} = \mathbf{R}_\omega \mathbf{R}_\phi \mathbf{R}_\kappa \qquad \ldots (2.2)$$

Matrix multiplication gives

$$\mathbf{A} = \begin{bmatrix} \cos\phi \cos\kappa & -\cos\phi \sin\kappa & \sin\phi \\ a_{21} & a_{22} & -\sin\omega \cos\phi \\ a_{31} & a_{32} & \cos\omega \cos\phi \end{bmatrix} \qquad \ldots (2.3)$$

with

$$a_{21} = \sin\omega \sin\phi \cos\kappa + \cos\omega \sin\kappa$$
$$a_{22} = -\sin\omega \sin\phi \sin\kappa + \cos\omega \cos\kappa$$
$$a_{31} = -\cos\omega \sin\phi \cos\kappa + \sin\omega \sin\kappa$$
$$a_{32} = \cos\omega \sin\phi \sin\kappa + \sin\omega \cos\kappa$$

The cameras of photogrammetric plotting instruments have mutually orthogonal axes. If the κ-axis is the tertiary axis, the κ-rotation is applied about a vertical Z-axis only if at that time the rotations ω and ϕ are equal to zero. If the ϕ-axis is the secondary axis and the ω-axis is the primary axis, a ϕ-rotation is applied about a horizontal axis only if at that time the ω-rotation is equal to zero. Therefore, the matrix in equation (2.2) represents the orientation matrix for an instrument with a tertiary κ-axis, a secondary ϕ-axis and a primary ω-axis. This matrix has been recommended for use in analytical photogrammetry in a resolution of the International Society for Photogrammetry, adopted at the 1960 Congress.

Each of the six possible arrangements of successive transformations $\mathbf{R}_\omega$, $\mathbf{R}_\phi$, and $\mathbf{R}_\kappa$ corresponds to one of the six possible choices of primary, secondary, and tertiary axes. Each leads to an orientation matrix in which one of the off-diagonal elements is equal to the sine of the secondary rotation.

Obviously, orthogonal matrices are obtained also if in any one of the matrices (2.1) the minus sign is attached to the other sine. This changes the positive direction of a rotation.

By changing the arrangement of the matrices and the position of the minus signs, 48 different proper orthogonal matrices can be constructed from three rotations. Each corresponds to a certain choice of primary, secondary and tertiary axis and of positive directions of the rotations.

Because of the one-to-one correspondence between the possible attitudes of a photograph and the proper orthogonal matrices of order three, for a given attitude of the photograph, these 48 matrices are numerically the same. The difference consists in the formulation of the elements as functions of three parameters and in the values of these parameters.

In the case of analytical aerial triangulation, where the camera axis at the moment of exposure is usually nearly vertical and the x-axis of each photograph can be chosen roughly parallel to the X-axis of the strip coordinate system, the three rotations in each of these matrices are only small. This makes these matrices suitable for use in the iterative procedure of relative orientation. From the mathematical point of view, they are all equally useful.

If in each of these matrices a minus sign is attached to the cosine of an angle, if sine and cosine of an angle are interchanged, or if more than one of these changes are introduced simultaneously, proper orthogonal matrices are obtained also. Geometrically, each of these changes means replacing a small angle by an angle which is closer to 90°, 180°, or 270°. Consequently, these matrices are less useful for analytical aerial triangulation.

Orthogonal matrices are obtained also if the algebraic construction starts with an on-diagonal element which is equated to the cosine or the sine of an angle. Geometrically, this corresponds to applying the first and third rotations about the same axis, and applying the second rotation about one of the other two axes. If the first and third rotations are applied about the Z-axis, these rotations are known as swing of the photograph and azimuth of the principal plane, respectively. These matrices are not useful here because, if the second rotation is equal to zero, the first and third rotations are not defined. Also, when the second rotation is not equal to zero, the first and third rotations may have any value from 0° to 360°.

3. Rotation about a directed line

Starting from the position where the x, y, z axes coincide with the X, Y, Z axes, the correct attitude of a photograph can be established by means of a single rotation about a suitable axis through the origin of the two coordinate systems.

This statement is obviously true if the orientation leaves at least one point, besides the origin, in its initial position. The axis of rotation is then the line through this point and the origin.

In other words, the statement is true if always a set of values of x, y, and z can be found, not all three equal to zero, for which $\mathbf{A}\,\mathbf{x} = \mathbf{x}$. According to a theorem of linear algebra, this is the case if and only if $|\mathbf{A} - \mathbf{I}| = 0$. It can be proved that this determinant is indeed equal to zero by writing it as a polynomial in the nine elements of $\mathbf{A}$ and simplifying this expression by means of first theorem III, then theorem I.

Let the matrix $\mathbf{A}$ have the direction cosines λ, μ, and ν of the axis of rotation and the angle of rotation α as parameters. Since

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

the matrix will again contain three independent parameters. In terms of these parameters, the matrix can be derived by means of a theorem from matrix algebra which states that the rotation about a directed line can be represented by the matrix

$$\mathbf{A} = \mathbf{T}\,\mathbf{R}_{\omega}\,\mathbf{T}^{-1} \qquad \ldots (3.1)$$

in which $\mathbf{T}$ is any proper orthogonal matrix with the direction cosines of the axis of rotation as the elements of the first column. Multiplication of the three matrices gives

$$\mathbf{A} = \begin{bmatrix} \lambda^2(1-\cos\alpha)+\cos\alpha & \lambda\mu(1-\cos\alpha)-\nu\sin\alpha & \lambda\nu(1-\cos\alpha)+\mu\sin\alpha \\ \lambda\mu(1-\cos\alpha)+\nu\sin\alpha & \mu^2(1-\cos\alpha)+\cos\alpha & \mu\nu(1-\cos\alpha)-\lambda\sin\alpha \\ \lambda\nu(1-\cos\alpha)-\mu\sin\alpha & \mu\nu(1-\cos\alpha)+\lambda\sin\alpha & \nu^2(1-\cos\alpha)+\cos\alpha \end{bmatrix} \qquad \ldots (3.2)$$

where $\lambda^2 + \mu^2 + \nu^2 = 1$.
This matrix can be found in a number of mathematical textbooks.

This form of the orientation matrix can be derived also by means of vector theory.

Let, as shown in Figure 6, $\overline{OQ}$ be the position vector $\mathbf{x}$ of a point before a rotation α about the axis OP and let $\overline{OR}$ be its position vector $\mathbf{X}$ after the rotation. The vector $\mathbf{X}$ can be regarded as the sum of a vector $\overline{OT}$ along the vector $\mathbf{x}$, a vector $\overline{TS}$ parallel to the axis of rotation, and a vector $\overline{SR}$ perpendicular to $\mathbf{x}$ and to the axis of rotation.

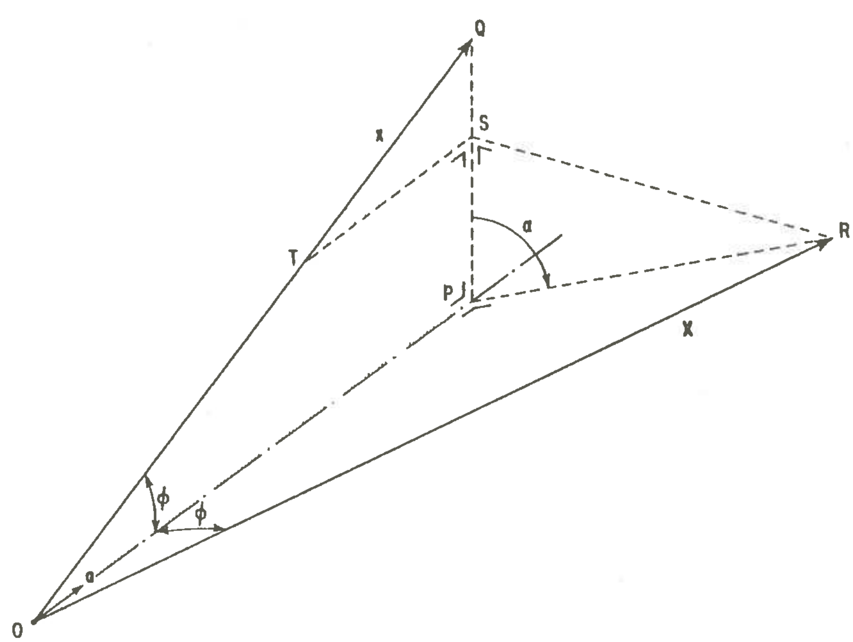

Figure 6. Rotation of a vector about a directed line.

Since the line element PS = PR cos α = PQ cos α, it follows that the line element OT = OQ cos α, and so

$$\overline{OT} = \mathbf{x}\cos\alpha\,. \qquad \ldots (3.3)$$

The dot product $\mathbf{a} \cdot \mathbf{x}$ of two vectors $\mathbf{a}$ and $\mathbf{x}$ is a scalar and is equal to the product of the lengths of the two vectors by the cosines of the angle between them. Therefore, if $\mathbf{a}$ is a unit vector along the axis OP, the length of the vector $\overline{OP}$ is $\mathbf{a} \cdot \mathbf{x}$ and so $\overline{OP} = (\mathbf{a} \cdot \mathbf{x})\mathbf{a}$ and

$$\overline{TS} = \overline{OP}(1 - \cos\alpha) = (\mathbf{a} \cdot \mathbf{x})\mathbf{a}(1 - \cos\alpha) \qquad \ldots (3.4)$$

The cross product $\mathbf{a} \times \mathbf{x}$ of the two vectors $\mathbf{a}$ and $\mathbf{x}$ is a vector parallel to and with the same positive direction as the vector $\overline{SR}$. Its length is the product of the lengths of the two vectors and the sine of the angle between them. Since the line element $SR = PR \sin\alpha = PQ \sin\alpha = OQ \sin\phi \sin\alpha$,

$$\overline{SR} = \mathbf{a} \times \mathbf{x} \sin\alpha \qquad \ldots (3.5)$$

Summation of equations (3.3), (3.4) and (3.5) gives

$$\mathbf{X} = \mathbf{x}\cos\alpha + \mathbf{a}(\mathbf{a} \cdot \mathbf{x})(1 - \cos\alpha) + \mathbf{a} \times \mathbf{x} \sin\alpha \qquad \ldots (3.6)$$

In this equation, the components of the vector $\mathbf{x}$ are the coordinates x, y, and z. The components of the unit vector $\mathbf{a}$ are the direction cosines λ, μ, and ν of the axis of rotation. Further, the dot product $\mathbf{a} \cdot \mathbf{x}$ is the sum of the products of the corresponding components:

$$\mathbf{a} \cdot \mathbf{x} = \lambda x + \mu y + \nu z \qquad \ldots (3.7)$$

The cross product $\mathbf{a} \times \mathbf{x}$ is a vector whose components are the cofactors of the elements of the first row of a matrix which has the components of $\mathbf{a}$ and of $\mathbf{x}$ as the elements of its second and third row, respectively:

$$\mathbf{a} \times \mathbf{x} = \begin{bmatrix} \mu z - \nu y \\ \nu x - \lambda z \\ \lambda y - \mu x \end{bmatrix} \qquad \ldots (3.8)$$

By substituting these expressions for the two vector products in equation (3.6), that equation can be written in terms of vector components. This shows that the equation is equivalent to equation (1.1), if $\mathbf{A}$ in that equation is the matrix of equation (3.2).

4. A purely algebraic derivation

The preceding derivations of orthogonal matrices were based upon the concept of the orientation of a photograph by means of one or more rotations. Although in the case of a matrix constructed from three rotations a construction was developed in which this concept was not directly employed, the interpretation of the parameters as being three rotations was always possible.

In analytical photogrammetry, however, there is no need for such an interpretation. It has the disadvantages of necessitating the computation of trigonometric functions. The following method allows the construction of the elements of the orthogonal matrix as rational functions of three independent parameters.

The skew-symmetric matrix with real elements

$$\mathbf{S} = \begin{bmatrix} o & -c & b \\ c & o & -a \\ -b & a & o \end{bmatrix} \qquad \ldots (4.1)$$

contains three independent parameters, as does the proper orthogonal matrix $\mathbf{A}$ of order three. Therefore, to each $\mathbf{S}$ corresponds one proper orthogonal matrix $\mathbf{A}$ with the same parameters.

A matrix $\mathbf{A}$ can be formulated in several ways as an analytical function of $\mathbf{S}$. Choosing a fourth parameter d, one can write

$$\mathbf{A} = (d\mathbf{I} + \mathbf{S})(d\mathbf{I} - \mathbf{S})^{-1} \qquad \ldots (4.2)$$

The matrix $(d\mathbf{I} - \mathbf{S})^{-1}$ can be found by computing the matrix of cofactors of $d\mathbf{I} - \mathbf{S}$, transposing this matrix, and dividing its elements by the determinant of $d\mathbf{I} - \mathbf{S}$. Alternatively, the elements of the inverse can be computed from the 3×3 equations contained in $(d\mathbf{I} - \mathbf{S})(d\mathbf{I} - \mathbf{S})^{-1} = \mathbf{I}$. Following this, multiplication of the two matrices in the right-hand side of equation (4.2) gives:

$$\mathbf{A} = \begin{bmatrix} d^2+a^2-b^2-c^2 & 2ab-2cd & 2ac+2bd \\ 2ab+2cd & d^2-a^2+b^2-c^2 & 2bc-2ad \\ 2ac-2bd & 2bc+2ad & d^2-a^2-b^2+c^2 \end{bmatrix} \frac{1}{d^2+a^2+b^2+c^2} \qquad \ldots (4.3)$$

This matrix satisfies the previously mentioned theorems I, II, III, and IV and is, therefore, proper orthogonal.

A different formulation of $\mathbf{A}$ as a function of $\mathbf{S}$ can be obtained by writing $\mathbf{A} = (\mathbf{A}^T)^{-1}$. Substituting the expression (4.2) into the second part of this equation gives:

$$\mathbf{A} = (d\mathbf{I} - \mathbf{S})^{-1}(d\mathbf{I} + \mathbf{S}) \qquad \ldots (4.4)$$

Since this equation is equivalent to (4.2), it also leads to the equation (4.3).

Multiplication of all four parameters by the same factor does not alter the value of the elements of $\mathbf{A}$. Therefore, $\mathbf{A}$ contains only three independent parameters.

Accordingly, it is possible to multiply the parameters by a factor which makes $d^2+a^2+b^2+c^2 = 1$ and makes d positive if it is negative. This reduces the matrix to the simpler but not less general form

$$\mathbf{A} = \begin{bmatrix} d^2+a^2-b^2-c^2 & 2ab-2cd & 2ac+2bd \\ 2ab+2cd & d^2-a^2+b^2-c^2 & 2bc-2ad \\ 2ac-2bd & 2bc+2ad & d^2-a^2-b^2+c^2 \end{bmatrix} \qquad \ldots (4.5)$$

in which $d^2+a^2+b^2+c^2 = 1$ and $d > 0$.

It is also possible to divide all four parameters by d. This simplifies the matrix to

$$\mathbf{A} = \begin{bmatrix} 1+a^2-b^2-c^2 & 2ab-2c & 2ac+2b \\ 2ab+2c & 1-a^2+b^2-c^2 & 2bc-2a \\ 2ac-2b & 2bc+2a & 1-a^2-b^2+c^2 \end{bmatrix} \frac{1}{1+a^2+b^2+c^2} \qquad \ldots (4.6)$$

This form of the orthogonal matrix can be obtained directly from

$$\mathbf{A} = (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}.$$

This formulation can be found in textbooks on linear algebra, usually with the signs interchanged.

This form appears to be the simpliest one for electronic computation and is the one which is used in the FORTRAN program.

IV. Relative Orientation

1. The elements of relative orientation

Let the strip triangulation be performed with respect to a three-dimensional rectangular coordinate system X, Y, Z as shown in Figure 7. For

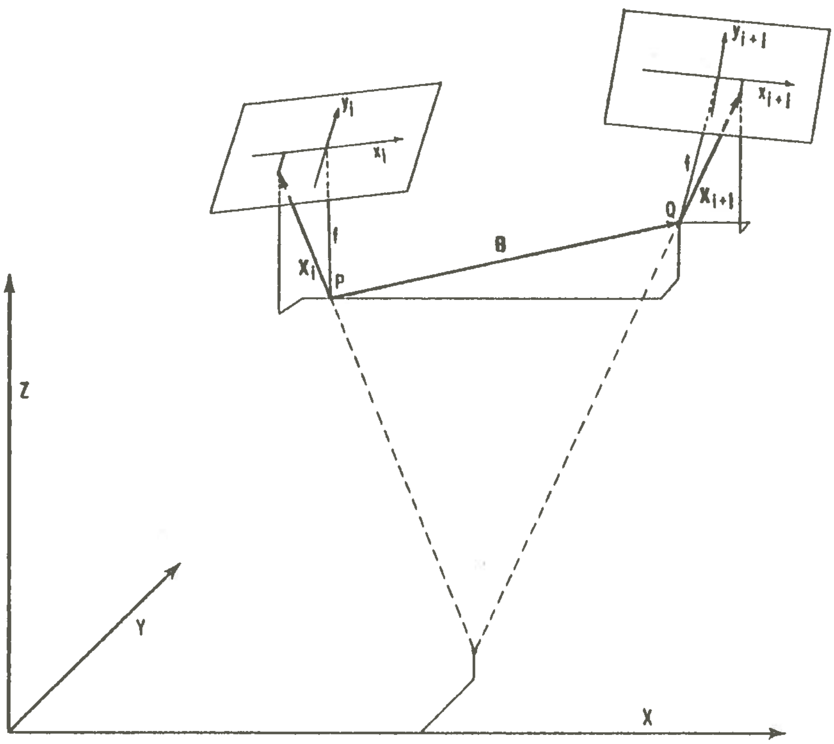

Figure 7. Vectors and vector components used in relative orientation.

each photograph, auxiliary coordinate systems X_i, Y_i, Z_i and x_i, y_i, z_i which have their origin in the projection centre will be required. These are the two systems used in the preceding chapter, with a subscript added to indicate the number of the photograph. Therefore, the X_i-, Y_i-, and Z_i-axes are parallel to the X-, Y-, and Z-axes and the x_i- and y_i-axes are parallel to the plane of the photograph.

The projection centre of the first photograph will be given arbitrary coordinates in the X, Y, Z system and the coordinate axes x_1, y_1, and z_1 of this photograph will be made to coincide with the X_1-, the Y_1-, and the Z_1-axis, respectively. This completes the orientation of the first photograph.

The strip triangulation will be performed by computing in succession the relative orientation of each following photograph with respect to the preceding one. Each relative orientation will be followed by scaling of the resulting model and by computation of the strip coordinates X, Y, and Z of the measured points.

To determine the relative orientation of each photograph with respect to the preceding one, an arbitrary value is assumed for the base component b_X while the x-, y-, and z-axes of the photograph are first placed parallel to the X-, Y-, and Z-axes. The elements of relative orientation are then the base components b_Y and b_Z and three independent parameters which determine the orientation matrix of the photograph.

2. The condition equation for relative orientation

The relative orientation of a photograph (i+1) with respect to the preceding one (i) consists in positioning the photograph in such a way that rays from corresponding images in the two photographs intersect.

Analytically, this means that a condition equation which states that corresponding rays intersect must be satisfied. The condition equation can state this requirement in different ways. For instance, it can state that the two rays must be co-planar, that the minimum distance between the rays must be equal to zero, or, assuming that the strip axis is approximately parallel to the x-axis, that the Y-parallaxes must be equal to zero.

The requirement of co-planarity can be formulated as the condition that two corresponding image points and the two projection centres must lie in one plane. According to an equation from analytical geometry, in this case a fourth-order determinant which has the strip coordinates of these points as the elements of the first three columns is equal to zero:

$$\begin{vmatrix} X^P & Y^P & Z^P & 1 \\ X^Q & Y^Q & Z^Q & 1 \\ X^P + X_i & Y^P + Y_i & Z^P + Z_i & 1 \\ X^Q + X_{i+1} & Y^Q + Y_{i+1} & Z^Q + Z_{i+1} & 1 \end{vmatrix} = 0 \qquad \ldots (2.1)$$

In this equation, the strip coordinates of the projection centres of photographs i and i+1 are indicated by superscripts P and Q.

From this equation it follows by subtraction of rows that

$$\begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ X_{i+1} & Y_{i+1} & Z_{i+1} \end{vmatrix} = 0 \qquad \ldots (2.2)$$

This is the condition equation for relative orientation. The base components b_X, b_Y, and b_Z in the first row of the determinant are the differences between the strip coordinates of the two projection centres. The subscripted coordinates in the second and third rows are the components of the vectors $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$

from projection centre to image point in photographs i and i+1. They are functions of the orientation matrices of the two photographs:

$$\mathbf{X}_i = \mathbf{A}_i \mathbf{x}_i$$
$$\mathbf{X}_{i+1} = \mathbf{A}_{i+1} \mathbf{x}_{i+1} \qquad \ldots (2.3)$$

Alternatively, the condition of co-planarity can state that the vectors $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ and the vector $\mathbf{B}$ from projection centre P to projection centre Q must lie in one plane. According to an equation from vector analysis, this requires that their scalar triple product must be equal to zero:

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} = 0 \qquad \ldots (2.4)$$

This is the condition equation for relative orientation in vector notation.

If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along X-, Y-, and Z-axis respectively, each of the three vectors in (2.4) can be written as the sum of vectors along these axes:

$$\begin{aligned} \mathbf{B} &= b_X \mathbf{i} + b_Y \mathbf{j} + b_Z \mathbf{k}, \\ \mathbf{X}_i &= X_i \mathbf{i} + Y_i \mathbf{j} + Z_i \mathbf{k}, \\ \mathbf{X}_{i+1} &= X_{i+1} \mathbf{i} + Y_{i+1} \mathbf{j} + Z_{i+1} \mathbf{k}. \end{aligned} \qquad \ldots (2.5)$$

With the help of equation (3.8) of the preceding chapter, the cross product $\mathbf{X}_i \times \mathbf{X}_{i+1}$ can now be written as a vector:

$$\mathbf{D} = (Y_i Z_{i+1} - Z_i Y_{i+1})\,\mathbf{i} + (Z_i X_{i+1} - X_i Z_{i+1})\,\mathbf{j} + (X_i Y_{i+1} - Y_i X_{i+1})\,\mathbf{k} \qquad (2.6)$$

With the help of equation (3.7) of that chapter, the dot product $\mathbf{B} \cdot \mathbf{D}$ can be written as a scalar. According to equation (2.4), this scalar is equal to zero:

$$b_X(Y_i Z_{i+1} - Z_i Y_{i+1}) + b_Y(Z_i X_{i+1} - X_i Z_{i+1}) + b_Z(X_i Y_{i+1} - Y_i X_{i+1}) = 0 \qquad \ldots (2.7)$$

Since this equation is obtained also if in (2.2) the determinant is expanded in terms of the elements of the first row, the equations (2.2) and (2.4) are equivalent. In the following sections, equation (2.4) will be used rather than equation (2.2) because it allows a more compact presentation of the formulas.

3. Differentiation of the condition equation

For each pair of corresponding image points, a condition equation (2.4) can be formulated. To compute the elements of relative orientation, at least five such equations obtained from five or more pairs of points must be available.

The equations are not linear with respect to the five elements. This makes it impossible to solve them directly on digital electronic computers. Most of these computers can perform no other mathematical operations than addition, subtraction, multiplication, and division.

Since linear equations can be solved by means of those operations, the condition equations must be replaced by linear approximations. Those can be derived by differentiating the condition equations with respect to the five orientation elements. The differentiation produces equations which are linear with respect to corrections to assumed approximations of the orientation elements.

Since the linear equations are only approximations of the condition equations, their solution gives only approximate values of the required corrections. Adding these corrections to the assumed approximations of the orientation elements gives improved approximations. Those must then be substituted into the linear equations and new corrections must be computed. The procedure must be repeated until the corrections become negligible. Thus, the orientation elements are computed in an iterative procedure.

Before equation (2.4) is differentiated, a modification can be introduced which results in simpler coefficients in the linear equation. This modification consists in premultiplying the matrix $\mathbf{A}_{i+1}$ by an orthogonal matrix $\mathbf{R}$. This changes the equation to

$$\mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{R}\mathbf{A}_{i+1}\mathbf{x}_{i+1}) = 0 \qquad \ldots (3.1)$$

The matrix $\mathbf{R}$ will be constructed from three parameters in the same way as the matrix $\mathbf{A}_{i+1}$ is constructed from its parameters. The matrix $\mathbf{A}_{i+1}$ will be the matrix of the assumed approximate orientation and the matrix $\mathbf{R}$ will serve to correct it. Thus, the parameters of $\mathbf{R}$ will be used as unknowns in the linear equations instead of corrections to the parameters of $\mathbf{A}_{i+1}$.

This equation must now be differentiated with respect to the components b_Y and b_Z of $\mathbf{B}$ and the three parameters of $\mathbf{R}$. The differentiation requires the use of approximate values for these five variables. Those for the two base components are equal to the assumed approximations and those for the three parameters are equal to zero.

Differentiation of $\mathbf{R}$ gives

$$\mathbf{R} \approx \mathbf{I} + \begin{bmatrix} o & -a_3 & a_2 \\ a_3 & o & -a_1 \\ -a_2 & a_1 & o \end{bmatrix} \qquad \ldots (3.2)$$

Here, a_1, a_2, and a_3 are functions of the parameters of the orthogonal matrix. For each of the suitable forms of the orthogonal matrix described in the preceding chapter, these functions are listed in Table 5.

If now $\mathbf{dB}$ is the vector whose components are the required corrections 0, db_Y, and db_Z to the base components and $\mathbf{R}_1$ is the skew-symmetric matrix in equation (3.2), the differentiation of equation (3.1) has as its result equation (3.4).

This equation can be obtained without using the rules for differentiation of a scalar triple product by linearization of equation (3.1). First, the

Table 5 Functions of the matrix parameters, to be solved from the condition equation (3. 1)

Matrix in the preceding chapter	Parameters a_1	a_2	a_3
(2. 3), derived from three rotations	ω	ϕ	κ
(3. 2), derived from one rotation	$\lambda\alpha$	$\mu\alpha$	$\nu\alpha$
(4. 5) and (4. 6), derived from a skew-symmetric matrix	2a	2b	2c

introduction of the correction vector $\mathbf{dB}$ and the linear approximation (3. 2) of $\mathbf{R}$ gives:

$$(\mathbf{B} + \mathbf{dB}) \cdot \mathbf{X}_i \times (\mathbf{X}_{i+1} + \mathbf{R}_1 \mathbf{X}_{i+1}) = 0 \qquad \ldots (3.3)$$

This equation still contains products of the corrections to the base components and the parameters of $\mathbf{R}$. It is linearized by writing the scalar triple product as a sum of such products and omitting the term which contains both $\mathbf{dB}$ and $\mathbf{R}_1$. This gives

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{dB} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{R}_1 \mathbf{X}_{i+1}) = 0 \qquad \ldots (3.4)$$

The product $\mathbf{R}_1 \mathbf{X}_{i+1}$ in this equation is a vector whose components can be obtained by performing the matrix multiplication. Its components are the same as those of the vector $\mathbf{r} \times \mathbf{X}_{i+1}$, where $\mathbf{r}$ is the vector with components a_1, a_2, and a_3, and therefore these two vectors are identical:

$$\mathbf{R}_1 \mathbf{X}_{i+1} = \begin{bmatrix} a_2 Z_{i+1} - a_3 Y_{i+1} \\ a_3 X_{i+1} - a_1 Z_{i+1} \\ a_1 Y_{i+1} - a_2 X_{i+1} \end{bmatrix} = \mathbf{r} \times \mathbf{X}_{i+1} \qquad \ldots (3.5)$$

Applied to equation (3.4), this gives

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{dB} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{r} \times \mathbf{X}_{i+1}) = 0 \qquad \ldots (3.6)$$

The value of a scalar triple product does not change if the dot and the cross are interchanged or if the factors of the dot product are interchanged. If the factors of the cross product are interchanged, the sign changes. This is immediately evident when the corresponding operations are performed upon the rows of the determinant with which the scalar triple product is identical.

By means of such changes and a rearrangement of the terms, the equation can be brought in its final form:

$$\mathbf{X}_{i+1} \times (\mathbf{B} \times \mathbf{X}_i) \cdot \mathbf{r} + \mathbf{X}_i \times \mathbf{X}_{i+1} \cdot \mathbf{dB} + \mathbf{X}_i \times \mathbf{X}_{i+1} \cdot \mathbf{B} = 0 \qquad \ldots (3.7)$$

This equation is obviously linear with respect to the five unknowns: the components of $\mathbf{r}$ and of $\mathbf{dB}$. Their coefficients are the components of the two vectors $\mathbf{X}_{i+1}\times(\mathbf{B}\times\mathbf{X}_i)$ and $\mathbf{X}_i\times\mathbf{X}_{i+1}$. These can be computed as such, and therefore the equation is in a suitable form for use in electronic computation. It is used in this form in the FORTRAN program. The third term is the constant part; it is equal to zero if the vectors from the projection centres to the image points intersect.

The equation can be written with a separate term for each of the five unknowns. For this purpose, the term with the vector $\mathbf{r}$ is written

$$\mathbf{B}\times\mathbf{X}_i\cdot(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times\mathbf{X}_{i+1}$$

According to the formula for a cross product

$$
\begin{aligned}
a_1\mathbf{i}\times\mathbf{X}_{i+1} &= (Y_{i+1}\mathbf{k} - Z_{i+1}\mathbf{j})\,a_1\\
a_2\mathbf{j}\times\mathbf{X}_{i+1} &= (Z_{i+1}\mathbf{i} - X_{i+1}\mathbf{k})\,a_2\\
a_3\mathbf{k}\times\mathbf{X}_{i+1} &= (X_{i+1}\mathbf{j} - Y_{i+1}\mathbf{i})\,a_3 \qquad \ldots (3.8)
\end{aligned}
$$

Further,

$$d\mathbf{B} = db_Y\,\mathbf{j} + db_Z\,\mathbf{k} \qquad \ldots (3.9)$$

If these expressions are introduced in the condition equation, it becomes:

$$
\begin{aligned}
&\mathbf{B}\times\mathbf{X}_i\cdot(Y_{i+1}\mathbf{k} - Z_{i+1}\mathbf{j})\,a_1 + \mathbf{B}\times\mathbf{X}_i\cdot(Z_{i+1}\mathbf{i} - X_{i+1}\mathbf{k})\,a_2 + \mathbf{B}\times\mathbf{X}_i\cdot(X_{i+1}\mathbf{j} - Y_{i+1}\mathbf{i})\,a_3\\
&\quad + (\mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{j})\,db_Y + (\mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{k})\,db_Z + \mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{B} = 0 \qquad \ldots (3.10)
\end{aligned}
$$

If now the scalar triple products are replaced by determinants and the corrections to the base components are added to the base components themselves, the condition equation is obtained in the form in which it appears in earlier publications of N.R.C.:

$$
\begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ 0 & -Z_{i+1} & Y_{i+1} \end{vmatrix} a_1 +
\begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ Z_{i+1} & 0 & -X_{i+1} \end{vmatrix} a_2 +
\begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ -Y_{i+1} & X_{i+1} & 0 \end{vmatrix} a_3 +
$$

$$
+ \begin{vmatrix} Z_i & X_i \\ Z_{i+1} & X_{i+1} \end{vmatrix} (b_Y + db_Y) +
\begin{vmatrix} X_i & Y_i \\ X_{i+1} & Y_{i+1} \end{vmatrix} (b_Z + db_Z) +
\begin{vmatrix} Y_i & Z_i \\ Y_{i+1} & Z_{i+1} \end{vmatrix} b_X = 0 \qquad \ldots (3.11)
$$

4. Differentiation with respect to the photograph coordinates

If five points have been measured for relative orientation, the unknowns can be solved from the resulting five linear equations (3.7).

In practice, both as a check on errors and to improve the accuracy of the relative orientation, more than five points will be measured. This makes an adjustment necessary. For this, the method of least squares provides a convenient algorithm. In order not to complicate the computer program, this algorithm can be used even if only five equations are available.

The method of least squares requires that each equation (3.7) be given the proper weight. Determination of the weight requires differentiation of the condition equation (2.4) not only with respect to the five unknowns but also with respect to the measured quantities, which are here the photograph coordinates.

The differentiation can be performed by first adding corrections dx_i, dy_i, dx_{i+1}, and dy_{i+1} to the photograph coordinates. Thus, vectors

$$\mathbf{dx}_i = dx_i\,\mathbf{i} + dy_i\,\mathbf{j}$$

and

$$\mathbf{dx}_{i+1} = dx_{i+1}\,\mathbf{i} + dy_{i+1}\,\mathbf{j} \qquad \ldots (4.1)$$

are added to the vectors $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, respectively, in equations (2.3). Subsequently, the obtained expressions for $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ are substituted into equation (3.3) and this equation is linearized in the same way as before.

This differentiation results in two additional terms in the correction equation (3.7):

$$\mathbf{B}\cdot(\mathbf{A}_i\,\mathbf{dx}_i)\times\mathbf{X}_{i+1} \quad \text{and} \quad \mathbf{B}\cdot\mathbf{X}_i\times(\mathbf{A}_{i+1}\,\mathbf{dx}_{i+1})$$

When now in these terms $\mathbf{dx}_i$ and $\mathbf{dx}_{i+1}$ are replaced by the expressions in the equations (4.1), the two terms can be changed into four terms, each of which contains one of the four corrections to the photograph coordinates. When simultaneously the terms are brought to the second part of the correction equation, this equation becomes:

$$\mathbf{X}_{i+1}\times(\mathbf{B}\times\mathbf{X}_i)\cdot\mathbf{r} + \mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{dB} + \mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{B} =$$
$$(\mathbf{B}\cdot\mathbf{X}_{i+1}\times(\mathbf{A}_i\,\mathbf{i}))\,dx_i + (\mathbf{B}\cdot\mathbf{X}_{i+1}\times(\mathbf{A}_i\,\mathbf{j}))\,dy_i +$$
$$+ (\mathbf{B}\cdot(\mathbf{A}_{i+1}\,\mathbf{i})\times\mathbf{X}_i)\,dx_{i+1} + (\mathbf{B}\cdot(\mathbf{A}_{i+1}\,\mathbf{j})\times\mathbf{X}_i)\,dy_{i+1} \qquad \ldots (4.2)$$

The coefficients of the four corrections are scalar triple products of three vectors. Matrix multiplication shows that the vector $\mathbf{A}_i\,\mathbf{i}$ in the first coefficient is the vector whose components are the elements of the first column of $\mathbf{A}_i$ and that the vector $\mathbf{A}_i\,\mathbf{j}$ in the second coefficient is the vector whose components are the elements of the second column of $\mathbf{A}_i$. Analogous rules apply to $\mathbf{A}_{i+1}\,\mathbf{i}$ and to $\mathbf{A}_{i+1}\,\mathbf{j}$.

The weight of each equation (4.2) is a function of the four scalar triple products and of accuracy and correlation of the four coordinates.

If no correlation exists between the photograph coordinates and if all have the same accuracy, the weight of an equation is proportional to the sum of the squares of the four scalar triple products. In the case of a strip of aerial photographs taken with an approximately vertical camera axis and of strip triangulation in the direction of the X-axis, the base components b_Y and b_Z will be small compared with b_X. Further, the x- and y-axes will be approximately parallel to the X- and Y-axes and, therefore, the diagonal elements of $\mathbf{A}_i$ and $\mathbf{A}_{i+1}$ will be approximately equal to unity and the off-diagonal elements will be small compared with unity. From this it follows that in each equation the coefficients of dx_i and dx_{i+1} will be approximately equal to zero and the coefficients of dy_i and dy_{i+1} will be approximately equal to $b_X f$. Consequently, in this case the weights are all approximately the same and may be made equal to unity.

In practice, the accuracy of the photograph coordinates depends upon the position of the points. An investigation of Hallert [20] gave the result that for a Wild RC8 camera approximately

$$m_x = m_y = k(1 + 7r^2), \qquad \ldots (4.3)$$

where m_x and m_y are the standard deviations of the x- and y-coordinates, k is a constant, and r is the distance from the point to the principal point, expressed in the focal length as unit of length. If this result is accepted, each equation should be given the weight

$$w = \sqrt{((1 + 7r_i^2)^2 + (1 + 7r_{i+1}^2)^2)} \qquad \ldots (4.4)$$

In this way, points near the principal points receive a weight that is about six times greater than the weight of points in the corners of a model.

5. Formation and solution of normal equations

Each point that is to be used to establish the relative orientation has now provided one linear equation (3.7). In the method of least squares, these equations are referred to as the correction equations. In matrix notation, they can together be represented by the equation

$$\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{o} \qquad \ldots (5.1)$$

Here, $\mathbf{A}$ is the matrix which has as the elements of each row the coefficients of one of the equations (3.7), $\mathbf{x}$ is the column vector whose components are the five unknowns, and $\mathbf{b}$ is the column vector whose components are the constant terms. Obviously, these notations have no connection with the earlier used symbols.

If more than five correction equations are available, they are in general inconsistent and no vector $\mathbf{x}$ exists that can satisfy the matrix equation (5.1). According to the method of least squares, the most probable value of $\mathbf{x}$ is then the value for which the quadratic term

$$(\mathbf{A}\mathbf{x} + \mathbf{b})^T \mathbf{W} (\mathbf{A}\mathbf{x} + \mathbf{b})$$

attains its minimum. Under the present assumptions of uncorrelated observations, $\mathbf{W}$ is a matrix whose diagonal elements are the weights attached to the correction equations and whose off-diagonal elements are equal to zero.

It can be proved that the quadratic form attains its minimum for that value of $\mathbf{x}$ which satisfies the matrix equation

$$\mathbf{A}^T\mathbf{W}\mathbf{A}\,\mathbf{x} = -\mathbf{A}^T\mathbf{W}\,\mathbf{b} \qquad \ldots (5.2)$$

This matrix equation comprises a set of five linear equations known as the normal equations.

The coefficients and the second parts of the normal equations could be computed by storing the complete matrix $\mathbf{A}$ and the vector $\mathbf{b}$ and by then performing the matrix multiplications $\mathbf{A}^T\mathbf{W}\mathbf{A}$ and $-\mathbf{A}^T\mathbf{W}\mathbf{b}$.

However, since here $\mathbf{W}$ is a diagonal matrix, the contribution of each correction equation to these matrix products can be computed separately. Let a correction equation be represented by the equation

$$\mathbf{a}_r\mathbf{x} + b = 0, \qquad \ldots (5.3)$$

where $\mathbf{a}_r$ is the row vector whose elements are the coefficients in the equation, $\mathbf{x}$ is again the column vector whose components are the five unknowns and b is the constant term. Let further $\mathbf{a}_c$ be the column vector which is the transpose of $\mathbf{a}_r$. For each correction equation, a matrix $w\,\mathbf{a}_c\,\mathbf{a}_r$ and a column vector $-wb\,\mathbf{a}_c$ can be computed. Here, w is the weight assigned to the equation. It can easily be shown that the matrix of coefficients and the vector of second parts in equation (5.2) are simply the sum of the matrices $w\,\mathbf{a}_c\,\mathbf{a}_r$ and the sum of the vectors $-wb\,\mathbf{a}_c$, respectively. In this way, the normal equations are computed as:

$$[w\,\mathbf{a}_c\,\mathbf{a}_r]\,\mathbf{x} = [-wb\,\mathbf{a}_c] \qquad \ldots (5.4)$$

The normal equations can be solved by Gaussian elimination and back substitution. In the elimination procedure, successive elements on the main diagonal of the matrix of coefficients can be used as pivotal elements.

6. Remarks on the derivations

i. Equation (3.5) proves that the vectors

$$\mathbf{R}_1\mathbf{X}_{i+1} \quad \text{and} \quad \mathbf{r} \times \mathbf{X}_{i+1}$$

are equivalent. The proof is somewhat inelegant because one of these two formulations could not be directly converted to the other. Instead, it has been shown that the two vectors have the same components.

If the components of $\mathbf{r}$ are identified with rotations about the X_{i+1}, Y_{i+1}, and Z_{i+1} axes, a direct conversion is possible. The change in the orientation, defined by the matrix $\mathbf{R}$, can be produced by three such rotations. If the three rotations are infinitesimal and are called a_1, a_2, and a_3, respectively, according to a theorem from vector analysis they change the vector $\mathbf{X}_{i+1}$ by vectors

$$a_1\mathbf{i} \times \mathbf{X}_{i+1},\; a_2\mathbf{j} \times \mathbf{X}_{i+1}, \text{ and } a_3\mathbf{k} \times \mathbf{X}_{i+1},$$

respectively. Therefore, since

$$\mathbf{r} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \ldots (6.1)$$

$\mathbf{X}_{i+1}$ is changed to $\mathbf{X}_{i+1} + \mathbf{r} \times \mathbf{X}_{i+1}$. This expression can immediately replace the expression between the brackets in equation (3. 1).

Although the three parameters have been defined as rotations, it is not necessary to interpret them as such when the orthogonal matrix $\mathbf{R}$ is computed. Instead, Table 5 can be used to change to a different set of parameters. However, a change in interpretation during the course of the derivations is not very elegant either.

ii. A second inelegant feature of the derivations is the use of matrix products as well as vector products.

In the individual equations, this mixed mode of formulation can be avoided by writing the condition of intersection as a matrix equation. Following Thompson [21, 22], the condition equation (2. 4) is first replaced by the equivalent expression

$$\mathbf{X}_i \cdot \mathbf{B} \times \mathbf{X}_{i+1} = 0 \qquad \ldots (6.\ 2)$$

According to equation (3. 5), the cross product in this equation represents the same vector as the matrix product $\mathbf{B}\mathbf{X}_{i+1}$, if in the latter $\mathbf{B}$ is the skew-symmetric matrix whose parameters are the components of the base vector. Also, the scalar product of $\mathbf{X}_i$ and $\mathbf{B}\mathbf{X}_{i+1}$ can be written as the matrix product of the row vector $\mathbf{X}_i$ and the column vector $\mathbf{B}\mathbf{X}_{i+1}$. This gives the matrix equation

$$\mathbf{X}_i^T \mathbf{B} \mathbf{X}_{i+1} = 0 \qquad \ldots (6.\ 3)$$

Introduction of the corrections to the base components and of the orthogonal matrix $\mathbf{R}$, and linearization of $\mathbf{R}$ leads to the equation

$$\mathbf{X}_i^T (\mathbf{B} + \mathbf{dB})(\mathbf{I} + \mathbf{R}_1)\mathbf{X}_{i+1} = 0 \qquad \ldots (6.\ 4)$$

which is equivalent to equation (3. 3). Linearization of this equation similar to that of equation (3. 3) gives

$$\mathbf{X}_i^T (\mathbf{B}\mathbf{R}_1 + \mathbf{dB})\mathbf{X}_{i+1} + \mathbf{X}_i^T \mathbf{B} \mathbf{X}_{i+1} = 0 \qquad \ldots (6.\ 5)$$

This equation is linear with respect to the five orientation elements. However, since each of these occurs in two elements of a matrix, the equation is unsuitable for use in electronic computation.

A return to vector algebra gives first

$$\mathbf{X}_i \cdot \mathbf{B} \times (\mathbf{r} \times \mathbf{X}_{i+1}) + \mathbf{X}_i \cdot \mathbf{dB} \times \mathbf{X}_{i+1} + \mathbf{X}_i \cdot \mathbf{B} \times \mathbf{X}_{i+1} = 0 \qquad \ldots (6.\ 6)$$

in which $\mathbf{B}$ and $\mathbf{dB}$ are again vectors. From this equation, equation (3. 7) follows immediately.

Alternatively, equation (6. 5) can be converted into equation (3. 11) by performing the matrix multiplications, collecting the terms with the same unknowns, and changing the signs of all terms.

Consequently, this derivation requires two changes of mathematical discipline, as well as the proof in equation (3.5). This is not an improvement.

It should be noted that for this derivation it is not necessary to replace equation (6.1) by equation (6.2). Instead of the base vector, the vector $\mathbf{X}_i$ can be replaced by a skew-symmetric matrix.

iii. An aesthetically satisfactory derivation of the correction equation can be obtained if vector analysis or tensor analysis is used exclusively. For this purpose, a third vector product, the dyad, and the sum of such products, the dyadic, must be used.

In the preceding sections of this chapter, these have not been used because they are relatively unknown. Here, they will be defined first and the required theorems will be listed.

The dyad $\mathbf{st}$, where $\mathbf{s}$ and $\mathbf{t}$ are vectors, is an operator which transforms a third vector $\mathbf{u}$ into another vector as follows:

$$\begin{aligned} \mathbf{st}\cdot\mathbf{u} &= \mathbf{s}(\mathbf{t}\cdot\mathbf{u}) \\ \text{and} \qquad \mathbf{u}\cdot\mathbf{st} &= (\mathbf{u}\cdot\mathbf{s})\mathbf{t} \end{aligned} \qquad \ldots (6.7)$$

Since $\mathbf{st}\cdot\mathbf{u}$ is a vector parallel to $\mathbf{s}$ and $\mathbf{ts}\cdot\mathbf{u}$ is a vector parallel to $\mathbf{t}$, the product $\mathbf{st}$ does not conform to the commutative law.

By definition, the distributive law applies to the product of a sum of dyads and a vector. Thus, the products $\mathbf{A}\cdot\mathbf{u}$ and $\mathbf{u}\cdot\mathbf{A}$ of a dyadic and a vector can be written as sums of vectors by first applying the distributive law and then using equation (6.7).

It follows from the definitions that this third vector product conforms to the distributive law. Therefore, by writing each vector in a dyadic as a linear function of three unit vectors, as in equation (6.1), using this distributive law, and collecting the resulting terms with the same dyad, any dyadic can be written in the form

$$\begin{aligned} \mathbf{A} = {} & a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} \\ & + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} \\ & + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk} \end{aligned} \qquad \ldots (6.8)$$

The dyadic is then said to be in its nonion form, and the nine coefficients are called its components.

In the following, dyadics will always be used in their nonion form.

The matrix $\mathbf{A}$ formed by the nine components is called the matrix of the dyadic. By writing each vector as a linear function of the unit vectors, as in equation (6.1), and expanding the products, it follows easily that the product $\mathbf{A}\cdot\mathbf{u}$ of a dyadic and a vector and the matrix product $\mathbf{A}\mathbf{u}$, where $\mathbf{A}$ is the matrix of the dyadic, represent the same vector.

The dot product of two dyadics $\mathbf{R}$ and $\mathbf{A}$ is the dyadic defined by

$$(\mathbf{R} \cdot \mathbf{A}) \cdot \mathbf{x} = \mathbf{R} \cdot (\mathbf{A} \cdot \mathbf{x}) \qquad \ldots (6.9)$$

It follows from this that the dot product of two dyads is

$$\mathbf{rs} \cdot \mathbf{uv} = (\mathbf{s} \cdot \mathbf{u})\,\mathbf{rv} \qquad \ldots (6.10)$$

Also, the components of the dyadic $\mathbf{R} \cdot \mathbf{A}$ are identical with the elements of the product of the matrices $\mathbf{R}$ and $\mathbf{A}$.

The cross product of a dyad and a vector is a dyad:

$$\mathbf{rs} \times \mathbf{u} = \mathbf{r}(\mathbf{s} \times \mathbf{u}) \qquad \ldots (6.11)$$

Through the distributive law, this definition is extended to the cross product of a dyadic and a vector.

It follows from the definitions that the following relation exists between any dyadic $\mathbf{A}$ and any two vectors $\mathbf{r}$ and $\mathbf{s}$:

$$(\mathbf{A} \times \mathbf{r}) \cdot \mathbf{s} = \mathbf{A} \cdot (\mathbf{r} \times \mathbf{s}) \qquad \ldots (6.12)$$

In terms of the unit vectors, the unit dyadic or idemfactor is:

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \qquad \ldots (6.13)$$

The dot product of the idemfactor and a vector is the same vector:

$$\mathbf{I} \cdot \mathbf{r} = \mathbf{r} \qquad \ldots (6.14)$$

Therefore, substitution of the idemfactor in equation (6.12) gives:

$$(\mathbf{I} \times \mathbf{r}) \cdot \mathbf{s} = \mathbf{r} \times \mathbf{s} \qquad \ldots (6.15)$$

The cross product $\mathbf{I} \times \mathbf{r}$ is in terms of the unit vectors:

$$\mathbf{I} \times \mathbf{r} = a_1(\mathbf{kj} - \mathbf{jk}) + a_2(\mathbf{ik} - \mathbf{ki}) + a_3(\mathbf{ji} - \mathbf{ij}) \qquad \ldots (6.16)$$

Such a dyadic, whose matrix is skew-symmetric, is called antisymmetric. Thus, equation (6.15) states the theorem that the cross product of two vectors is equal to the dot product of the antisymmetric dyadic, formed from the first vector, and the second vector.

Analogous theorems can be formulated when tensor analysis is used. The vectors and dyadics are then replaced by tensors of valence one and two, respectively.

With the help of these theorems, the condition equation for relative orientation can now be linearized. The equation itself remains unchanged:

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} = 0 \qquad \ldots (6.17)$$

Dyadics $\mathbf{A}_{i+1}$ and $\mathbf{R}$ are now introduced to define the approximate orientation of photograph i+1 and its correction, respectively. The dyadics will be used in their nonion form, and their matrices will be defined as being the matrices $\mathbf{A}_{i+1}$ and $\mathbf{R}$ of the preceding sections.

This brings the condition equation in a form which is equivalent to equation (3.1):

$$\mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{R} \cdot (\mathbf{A}_{i+1} \cdot \mathbf{x}_{i+1})) = 0 \qquad \ldots (6.18)$$

Once $\mathbf{R}$ has been computed, the dyadic $\mathbf{R} \cdot \mathbf{A}_{i+1}$ will become the new approximation or the final value of $\mathbf{A}_{i+1}$. The components of this dyadic are, according to equation (6.9), identical with the elements of the matrix $\mathbf{R}\mathbf{A}_{i+1}$.

The equation (6.18) is linearized by differentiation. The differentiation follows the ordinary rules for the differentiation of a product. The result is an equation which is equivalent to equation (3.4):

$$\mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{R}_1 \cdot \mathbf{X}_{i+1}) + d\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} = 0 \quad \ldots (6.19)$$

Here, $\mathbf{R}_1$ is the antisymmetric dyadic formed from the parameters of $\mathbf{R}$. Therefore

$$\mathbf{R}_1 = \mathbf{I} \times \mathbf{r} \;, \qquad \ldots (6.20)$$

where $\mathbf{r}$ is the vector of equation (6.1). By means of this equation and the theorem in equation (6.15), equation (6.19) can now be written

$$\mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{r} \times \mathbf{X}_{i+1}) + d\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} = 0 \quad \ldots (6.21)$$

This equation is identical with equation (3.6). It is converted to equation (3.7) by the method described in section 3.

7. Remarks on the computations

i. Simplification of the condition equation

When a strip of aerial photographs is triangulated, the matrix $\mathbf{A}_i$ which is used in the correction equation for relative orientation can be the orientation matrix of photograph i computed during the relative orientation of that photograph. The best first approximation of the orientation matrix of photograph i+1 will then be the same matrix and the best first approximation of the base components b_Y and b_Z will be the definitive values of the base ratios in the preceding model. This method was used in the NRC program for the IBM 650.

However, it is also possible to replace the matrix $\mathbf{A}_i$ in the correction equation by the unit matrix. The first approximation of the matrix $\mathbf{A}_{i+1}$ will then also be the unit matrix, and the first approximation of b_Y and b_Z will be equal to zero. After completion of the relative orientation, the obtained matrix $\mathbf{A}_{i+1}$ and the base must then be premultiplied by the matrix $\mathbf{A}_i$. This method has been used in the FORTRAN program.

Neither method has great advantages over the other. The change from one method to the other was made mainly because the FORTRAN program for the IBM 1620 was originally intended to be a program for single models only.

When the second method is used, the correction equation for the first iteration of the relative orientation becomes much simpler. If in addition the photograph coordinates are divided by the focal length and the base component b_X is made equal to unity, the vectors $\mathbf{B}$, $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ become:

$$\mathbf{B} = \mathbf{i}$$
$$\mathbf{X}_i = x_i\,\mathbf{i} + y_i\,\mathbf{j} + \mathbf{k}$$
$$\mathbf{X}_{i+1} = x_{i+1}\,\mathbf{i} + y_{i+1}\,\mathbf{j} + \mathbf{k} \qquad \ldots (7.1)$$

Substitution of these vectors into equation (3. 7) or of their components into equation (3. 11) gives the simple correction equation

$$(y_i y_{i+1} + 1)a_1 - x_{i+1}y_i(a_2 + b_Z) - x_{i+1}a_3 + (x_{i+1} - x_i)b_Y - x_i y_{i+1} b_Z + (y_i - y_{i+1}) = 0 \qquad \ldots (7.2)$$

in which x_i, y_i, x_{i+1}, and y_{i+1} are the photograph coordinates after division by the focal length.

ii. Simplification of the matrix $\mathbf{R}$

In the FORTRAN program, equation (4. 6) of chapter III is used to compute the correction matrix $\mathbf{R}$ from its three parameters.

If the division of each element of this matrix by their common denominator were omitted the matrix and, as a result, the transformed vectors X_{i+1} would be multiplied by the factor $1 + a^2 + b^2 + c^2$.

Since a comparison of equations (3. 2) and (4. 6) in chapter III shows that the relations between the two sets of parameters in these equations are:

$$a = \lambda \tan \tfrac{1}{2}\alpha$$
$$b = \mu \tan \tfrac{1}{2}\alpha$$
$$c = \nu \tan \tfrac{1}{2}\alpha \qquad \ldots (7.3)$$

it follows that

$$1 + a^2 + b^2 + c^2 = 1 + \tan^2 \tfrac{1}{2}\alpha \qquad \ldots (7.4)$$

Even if the convergence of two photographs is 90° ($\alpha = 90°$, $b = \mu = 1$, $a = \lambda = c = \nu = 0$), this factor is only equal to 2. If the difference in tilt is 5°, it is equal to 1.002.

Therefore, if the division were omitted, the diagonal elements of the matrix $\mathbf{R}$ could become somewhat larger than unity. Because of the use of floating-point arithmetic, this will cause inaccurate values of the least significant digit of the elements of $\mathbf{A}_{i+1}$. If the length of the mantissas of the floating-point numbers is 10 decimal digits or more, this will have no effect upon the computed strip coordinates.

V. Absolute orientation and computation of strip coordinates

1. Absolute orientation

During the computation of the relative orientation, the base component b_X is assigned unit length.

The scaling of the model is now performed by computing the scale factor which reduces the model to the proper scale and by multiplying the three base components by it.

The first model of a strip can be given an arbitrary scale. This can be done by specifying the length of the base component b_X or of the base $\mathbf{B}$. It is of advantage to specify that this length in the triangulated strip must be the same as the length in the photographs. In this case the triangulated strip will have the same scale as the photographs and the quality of the triangulation will immediately be evident from the size of the residual parallaxes and from the coordinate differences of points that are common to two models.

The scale of all other models must be reduced to that of the first model. For this purpose, one or more distances in the model are compared with the same distances in the preceding, already scaled, model. The ratio of the distances is accepted as the required scale factor and, therefore, as the numerical value of the base component b_X.

This computation presents no problem if a scale transfer point in the preceding model and the same point in the present model lie on the same ray through the projection centre of the common photograph. The required scale factor is then simply the ratio between the distances from the point to the common projection centre in the two models. This ratio is then equal to the ratio between the differences in Z-coordinates of the point and the projection centre in the two models.

However, in general the point will not lie on the same ray in the two models. This is a result of the facts, discussed in the next section, that after the adjustment of the relative orientation corresponding rays do not in general intersect and that the point that is defined as the point of intersection will not lie on either of them. In addition, if the readings have been made on a stereocomparator, the two sets of readings of the point in the common photograph may differ slightly and, therefore, may produce slightly different rays.

As a result, if a point lies at some distance from the axis of the strip, the two vectors from the common projection centre to a scale transfer point can have slightly different transversal tilts in the two models. In that case, the distances from the point to the projection centre and the heights of the point in the two models cannot simultaneously be the same. Therefore, scaling with these distances and scaling with these heights will give slightly different results.

Since want of intersection of corresponding rays is caused by uncorrected errors, the origin of these errors is presumably unknown and it is difficult to say which of the two methods of determining the scale will serve best to reduce the effect of these errors upon the triangulation.

In the above case, where the two sets of readings of the point in the common photograph differ slightly, the points measured in the two models are not precisely the same. It is then more likely that the two measured points have the same terrain height than that they are at the same distance from the common exposure station. Therefore, it is better to use the heights for the scaling than to use the distances to the common projection centre.

In the FORTRAN program, the scale factor is derived as the ratio between the distances from the points to the plane $z = 0$ of the common photograph. This is equivalent to using the heights of the points in a coordinate system in which the common photograph has tilts equal to zero. Therefore, the computed scale factor is independent of the tilt of the strip in the X, Y, Z coordinate system.

The distance from a point to the plane $z = 0$ is numerically equal to the z-component of the vector from the projection centre to the point. Therefore, if $\mathbf{A}_i$ is the definitive orientation matrix of the common photograph and the subscript P refers to the common projection centre, these distances d_1 and d_2 in the preceding and the present model are:

$$d_1 = (\mathbf{X} - \mathbf{X}_P) \cdot \mathbf{A}_i \mathbf{k}$$

$$d_2 = (\mathbf{X} - \mathbf{X}_P) \cdot \mathbf{k} \qquad \ldots (1.1)$$

Here, $\mathbf{X}$ in the first equation is the definitive position vector of the point in the preceding model and $\mathbf{X}$ in the second equation is the position vector in the present model before absolute orientation. The second equation is simpler than the first one because in the present model the orientation matrix of the common photograph is still the unit matrix.

The scale factor is now computed as the mean of the ratios d_1/d_2 for all scale transfer points, and the base vector is multiplied by this factor.

The absolute orientation is completed by pre-multiplying the base and the orientation matrix of the new photograph by the definitive orientation matrix of the common photograph:

$$\mathbf{B} = (d_1/d_2)_{mean}\, \mathbf{A}_i (\mathbf{B})_{rel.\ or.}$$

$$\mathbf{A}_{i+1} = \mathbf{A}_i (\mathbf{A}_{i+1})_{rel.\ or.} \qquad \ldots (1.2)$$

The position vector of the projection centre Q of the new photograph is computed by adding the base vector to the position vector of the projection centre P of the common photograph:

$$\mathbf{X}_Q = \mathbf{X}_P + \mathbf{B} \qquad \ldots (1.3)$$

2. The point of intersection of two rays

The strip coordinates of all measured points can now be computed by intersecting the rays from corresponding image points in the two photographs.

If the photograph coordinates are free from all errors, the relative orientation is also error-free and each two such rays intersect. In practice that is not the case and consequently the rays cross at a short distance.

This makes it necessary to select a point that is to represent the point of intersection.

There are three points that can be used as such. The selection that is made will depend upon the point of view.

i. If the use of the method of least squares in the adjustment of the relative orientation is considered to be the best procedure, and not only a convenient one, it is logical to use this method also to give the photograph coordinates such corrections that each two rays do intersect. The point of intersection of the corrected rays will then be used.

This adjustment can be based upon equation (4.2) in chapter IV and it can be applied not only to the points which have been used to establish the relative orientation but also to all other points. Since orientation corrections are not computed at this stage, the terms with $\mathbf{r}$ and with $\mathbf{dB}$ are omitted from the equation. Representing further the constant term and the coefficients of the remaining terms by d, d_1, d_2, d_3, and d_4, the equation becomes

$$d_1 dx_i + d_2 dy_i + d_3 dx_{i+1} + d_4 dy_{i+1} = d \qquad \ldots (2.1)$$

Assuming equal accuracy of the coordinates and freedom from correlation, the application of the method of least squares to this equation gives the following corrections to the photograph coordinates:

$$\begin{aligned} dx_i &= d_1 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \\ dy_i &= d_2 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \\ dx_{i+1} &= d_3 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \\ dy_{i+1} &= d_4 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \end{aligned} \qquad \ldots (2.2)$$

ii. If the strip triangulation is performed on a first-order plotter, the point which represents the point of intersection is defined as a point in the horizontal plane at the height where the X-parallax between the two rays is equal to zero. It lies on the line which connects the points of intersection of the rays and this plane, midway between these points.

This point is obtained also if the first method of correcting the photograph coordinates is applied to exactly vertical photographs. This is the result of the fact that in this case, with the x_i - and x_{i+1} -axes parallel to the X-axis, the coefficients d_1 and d_3 become equal to zero while the coefficients d_2 and d_4 become equal. Since these coefficients change very little when the photographs are given small tilts, this choice of point is suitable for all nearly vertical photographs.

iii. If one wishes to represent the point of intersection by the point that lies as close to the two rays as possible, the point which lies midway on the line of shortest distance of the non-intersecting rays should be selected.

If the photographs are of a good quality the want of intersection of corresponding rays is small and consequently the three points are almost identical. If systematic errors occur, the method of least squares should be looked upon as only a convenient adjustment procedure. It is then impossible to say which point represents the point of intersection best. Therefore, in practice each of the three points is acceptable.

In the NRC program for the IBM 650, the second choice was made. In the present FORTRAN program, the third point has been chosen. However, a return to the second point would only require changing a few FORTRAN statements.

3. Computation of the point of intersection

In case ii and in case iii of the preceding section, the "point of intersection" of two corresponding rays lies on a specified line from a point on one of the rays to a point on the other ray, and midway between those two points.

This situation is shown in Figure 8, in which $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ are the vectors from the projection centres to the image points in the oriented photographs and

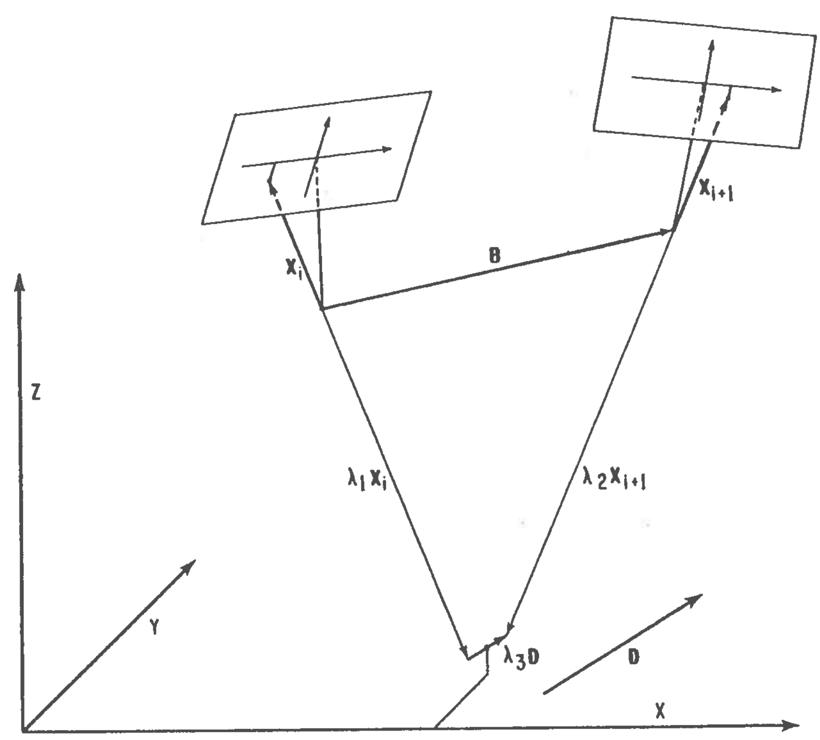

Figure 8. Definition of the position of a point in the case of non-intersecting rays.

$\mathbf{D}$ is a vector of arbitrary length, parallel to the line on which the point of intersection lies.

The two vectors from the projection centres to the above two points are collinear with $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ and, therefore, can be denoted by $\lambda_1 \mathbf{X}_i$ and $\lambda_2 \mathbf{X}_{i+1}$, in which the factors λ_1 and λ_2 are scalars. Similarly, the vector which connects the two points is parallel to $\mathbf{D}$ and can be denoted by $\lambda_3 \mathbf{D}$.

It follows directly from Figure 8 that

$$\mathbf{B} = \lambda_1 \mathbf{X}_i - \lambda_2 \mathbf{X}_{i+1} + \lambda_3 \mathbf{D} \qquad \ldots (3.1)$$

According to a theorem from vector analysis, any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ in three-dimensional space are connected by the equation

$$(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\mathbf{d} - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d})\mathbf{a} + (\mathbf{c} \cdot \mathbf{d} \times \mathbf{a})\mathbf{b} - (\mathbf{d} \cdot \mathbf{a} \times \mathbf{b})\mathbf{c} = 0 \quad (3.2)$$

This theorem is derived by regarding a vector product of these vectors in two ways as a triple vector product, expanding it and equating the two results:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d} \\ &= (\mathbf{a} \cdot \mathbf{c} \times \mathbf{d})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d})\mathbf{a} \qquad \ldots (3.3)\end{aligned}$$

Since equation (3.1) is valid only for unique values of λ_1, λ_2, and λ_3, and equation (3.2) is valid for any four vectors, the values of λ_1, λ_2, and λ_3 follow immediately when the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are equated with the vectors $\mathbf{B}$, $\mathbf{X}_i$, $\mathbf{X}_{i+1}$, and $\mathbf{D}$, and the coefficients of the two equations are compared. This gives

$$\lambda_1 = \frac{\mathbf{D} \cdot \mathbf{B} \times \mathbf{X}_{i+1}}{\mathbf{D} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}}, \qquad \lambda_2 = \frac{\mathbf{D} \cdot \mathbf{B} \times \mathbf{X}_i}{\mathbf{D} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}}, \qquad \lambda_3 = \frac{\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}}{\mathbf{D} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}} \quad \ldots (3.4)$$

Alternatively, but less elegantly, this result can be obtained by decomposing equation (3.1) into three equations between vector components, solving these equations by means of Cramer's rule and replacing the resulting determinants by the scalar triple products to which they are equivalent.

The position vector of the point in the strip coordinate system can now be computed by means of the equation

$$\mathbf{X} = \mathbf{X}_P + \lambda_1 \mathbf{X}_i + 0.5\, \lambda_3 \mathbf{D} \qquad \ldots (3.5)$$

A measure of the want of intersection of the two rays is the length of the vector $\lambda_3 \mathbf{D}$:

$$\text{Want of intersection} = \lambda_3 \sqrt{(\mathbf{D} \cdot \mathbf{D})} \qquad \ldots (3.6)$$

The above formulas can be especially adapted to each of the three cases in the preceding section.

In case i, the rays intersect after correction of the photograph coordinates. As a result, λ_3 is equal to zero and any vector which is not parallel to the plane which contains the base and the corrected image points can serve as the vector $\mathbf{D}$. A simple set of equations is obtained by taking for $\mathbf{D}$ the unit vector $\mathbf{j}$ in the positive Y-direction.

In case ii, the vector $\mathbf{D}$ must be parallel to the Y-axis. Here also, the unit vector $\mathbf{j}$ in the positive Y-direction is a suitable choice. This gives for λ_1 and λ_2 the simple expressions

$$\lambda_1 = \frac{b_Z X_{i+1} - b_X Z_{i+1}}{Z_i X_{i+1} - X_i Z_{i+1}} \quad \text{and} \quad \lambda_2 = \frac{b_Z X_i - b_X Z_i}{Z_i X_{i+1} - X_i Z_{i+1}} \qquad \ldots (3.7)$$

The X- and Z-coordinates of the point are now computed as the X- and Z-components of the vector $\mathbf{X}_P + \lambda_1 \mathbf{X}_i$ or, which gives the same result, those of the vector $\mathbf{X}_Q + \lambda_2 \mathbf{X}_{i+1}$. The Y-coordinate is computed as the mean of the Y-components of these two vectors and the want of intersection is computed as the difference.

In case iii, the vector $\mathbf{D}$ must be parallel to the line of shortest distance. The simplest choice, and the one used in the FORTRAN program, is

$$\mathbf{D} = \mathbf{X}_i \times \mathbf{X}_{i+1} \qquad \ldots (3.8)$$

This vector product is here substituted for $\mathbf{D}$ in the equations (3.4), (3.5), and (3.6). The want of intersection is positive if, in the area where the rays cross, points on the ray from photograph i+1 have greater Y-coordinates than those on the ray from photograph i, and it is negative if the former points have smaller Y-coordinates.

The computation of the position vectors of all measured points and of the want of intersection completes the computations for the model.

VI. A FORTRAN IV program

1. General remarks

This FORTRAN program follows the specifications described in the preceding chapters. It differs from the earlier NRC program for the IBM 650 and the IBM 1620 [2] in the following respects:

i. The photograph coordinates can be corrected for differential film shrinkage.

ii. The lens correction table contains a list of radial corrections for lens distortion instead of ratios of correction and radial distance.

iii. The corrections for earth curvature and refraction are applied directly by the program instead of being included in the table.

iv. The relative orientation of each photograph is started with the unit matrix as the orientation matrix of the preceding photograph and with base components b_Y and b_Z equal to zero.

v. The elements of the orientation matrix are computed as rational functions of the three parameters.

vi. For relative orientation, any number of points may be used. At the most, three iterations are performed.

vii. An experimental formula for weighting the equations for relative orientation is included as an option.

viii. The point of intersection of corresponding rays is defined as the point midway on their line of shortest distance. The want of intersection of corresponding rays is defined as their shortest distance.

ix. For scaling a model, a maximum of ten points may be used. These points are given equal weights.

x. The scaling is performed by comparing for each scale transfer point its distance in the present and in the preceding model to a plane which contains the common projection centre and is parallel to the plane of the common photograph.

The program was first written in FORTRAN II for use on a 40000-digit IBM 1620, in anticipation of the arrival of an IBM System/360 computer at the NRC laboratories in July 1965.

The small storage capacity of this IBM 1620 made it imperative to use as little space as possible for the instructions. For this reason, a great number of unsubscripted variables were used in the FORTRAN statements. They were equivalenced with elements of arrays by means of the EQUIVALENCE and COMMON statements. Also, the use of FORTRAN II for the IBM 1620 made it necessary to avoid mixed-mode expressions.

The program was subsequently converted to FORTRAN IV for use on the IBM System/360. The above features of the program have been retained in the FORTRAN IV program. With very minor modifications, this program has been in use since August 1965.

This chapter describes the December 1966 version of the FORTRAN IV program.

2. The iterative procedure of relative orientation

In chapter IV, the condition equation for relative orientation has been linearized and it has been shown how the linear equation can be used in an iterative procedure to determine the relative orientation.

The FORTRAN program performs at the most three iterations of the relative orientation.

In the first iteration, the unit matrix is used as the matrix of the approximate orientation of photograph i+1, and the approximations of the base components b_Y and b_Z are assumed to be equal to zero. Accordingly, the equation (IV 7. 2) is used as correction equation. In the second and in the third iteration, equation (IV 3. 7) is used.

If none of the five orientation parameters computed during the first iteration is larger than 1/30th of a radian (about 2°), only two iterations are performed; otherwise, three iterations are performed. If these parameters are all smaller than this test value, it is certain that the corrections which further iterations will apply will be of the order of 10' or smaller. Such corrections can be obtained by one iteration and, accordingly, a third iteration would then be superfluous.

In the first iteration and also in the second iteration, provided that this is not the final one, the FORTRAN program applies the weight 1 to each correction equation and it does not use other points than those whose coordinates can be stored in the arrays in core storage. In the final iteration, either each correction equation receives the weight 1 or its weight is computed by means of equation (IV 4. 4). Also, there is here no limit to the number of points that can be used.

If one wishes, the above test value can be changed by changing the constant in FORTRAN statement S. 0087.

3. The scaling of models

The scale of the triangulated strip is determined by the value of the base component b_X of the first model. This value is punched in the first card of the input deck for the strip and is expressed in microns at photograph scale. Therefore, if this value is equal to the actual length at photograph scale and the base components b_Y and b_Z are small, the computed strip coordinates and parallaxes will be expressed also, at least approximately, in microns at photograph scale.

For the scaling of the first model, only the base component b_X is required. Each following model is scaled to the preceding one by means of points in the overlap of the two models. At the most, ten points can be used and in each model these points must be among the first 100 of the points used for relative orientation. They are given equal weights.

The selection of points for scaling can be made either by specifying that one of four standard patterns is to be used or by marking the input cards of the selected points individually.

The standard patterns are selected by punching one of the digits 1, 2, 3, and 4 in the first field in the first data card of the strip. These patterns are the following (the numbering of the points refers to their sequence in the card deck and not to the point number).

1. Point 2 in one model is the same point as point 5 in the preceding model.
2. Points 2 and 3 in one model are the same points as points 6 and 7, respectively, in the preceding model.
3. Points 1, 2, and 3 in one model are the same points as points 4, 5, and 6, respectively, in the preceding model.
4. Points 1, 2, 3, and 4 in one model are the same points as points 5, 6, 7, and 8, respectively, in the preceding model.

Patterns 1 and 3 can be used when six of the points for relative orientation have been measured in the classical six positions. Patterns 2 and 4 can be used when eight of these points have been measured in the same positions: one in each corner of the model and two near each principal point. Patterns 1 and 2 serve to scale on points near the principal points only; patterns 3 and 4 make use of points near the strip edges also.

If in any model one or more points for relative orientation are marked by the digit 1 punched in column 40, these points will be used for scaling the following model. In such a case, a standard pattern will not be used even if it is specified on the first input card. Since the digit 1 is read as the number 1 in columns 38-40, if necessary it can be erased simply by punching a minus sign or a non-zero digit in column 39.

If a standard pattern is used, a point that is used for scaling need not have the same point number in the two models that are involved. If the 1-punch in column 40 is used, such a point must have the same point number in the two models.

The program will discard a scale transfer point if the difference between the scale factor derived from it and the mean of the scale factors from all as yet not discarded points is larger than 0.0005 times the mean. This elimination of anomalous points is performed in an iterative procedure, one point at a time. If here the scale factors of two anomalous points differ by the same amount from the mean, the second point is discarded.

The above test value can be changed by changing the constant in the FORTRAN statement S. 0155.

4. Input

The program uses a card reader as its input device.

Each deck of cards for the triangulation of a strip must contain all the information that is needed to perform the triangulation.

Decks for different strips, including single models, may be stacked for sequential processing. In this case, they must be separated from each other by means of a card with a negative non-zero number in its first four columns. A blank card must be placed behind the deck for a single strip triangulation and behind the last deck of a stack.

All data punched in the cards must be in the form required by FORTRAN fixed-point format: the least significant digit of each number must be punched in the right-most column of its field, and if a number is negative a minus sign must be punched in one of the columns of its field to the left of the most significant digit.

i. Cards with general information

The first few cards in the deck of a strip contain all the necessary information other than the measured coordinates.

The first card in the deck contains:

field 1, columns 1-4: The code number of the pattern for the scaling of the models. The code number is zero, and may be omitted, if the scaling is to be performed on points marked by a digit 1 in column 40 of the coordinate card in the preceding model. The code number is 1, 2, 3, or 4, respectively, if the scaling is to be performed on points in one of the four standard patterns described in the preceding section of this chapter.

field 2, columns 5-9: The code number for weighting the correction equations for relative orientation. The code number is zero, and may be omitted, if the equations of all points are to be given equal weights. It is 1, or any other positive non-zero number, if the experimental formula for Wild Aviogon 6" photography is to be used.

field 3, columns 10-16: The calibrated focal length in microns.

field 4, columns 17-23 and
field 5, columns 24-30: Correction factors for film shrinkage in x- and y-directions, respectively, multiplied by 100000.

field 6, columns 31-37: The value of the base component b_x of the first model, expressed in microns at photograph scale.

field 7, columns 38-44: The average flying height above ground, in meters. This value will be used for the computation of the earth curvature correction. If this correction must not be applied, this field must contain zeros or blanks.

field 8, columns 45-51: The coefficient c_1 in the formula for the refraction correction, multiplied by 10^7. This is the value of the photogrammetric refraction in Table II for the actual flying height and the average terrain height, multiplied by 10. If this correction must not be applied, this field must contain zeros or blanks.
Thus, the correction for earth curvature and the refraction correction will be used to correct the photograph coordinates of all points, irrespective of differences in terrain height.

field 9, columns 52-58: If no card output is needed, a non-zero number. Suppression of card output reduces the computer time and may be acceptable during the first triangulation of a strip.

columns 79 and 80: A serial number. This serial number provides the only check on the correct sequence of the cards with general information. Consecutive cards must be punched with consecutive numbers.

The second card and, if needed, following cards contain the lens correction table. This table consists of a list of values of the radial lens correction dr for values of the radial distance r which are separated by a constant interval and start at r = 0. The interval must be sufficiently small to allow linear interpolation between consecutive table values.

The second card is punched as follows:

field 1: The number of entries in the table. This number is at the most 162.

field 2: The interval of the argument r, expressed in 0.1 mm as unit of length.

fields 3 to 11, of 7 columns each, and covering columns 10 to 72: Up to nine values of the lens correction dr, starting with dr = 0 for r = 0, and followed by the values for consecutive values of r. The lens corrections are expressed in 0.01 micron as unit of length.

columns 79 and 80: The serial number of the card.

If more than nine values of the lens correction are to be punched, the remainder are punched sequentially in one or more following cards. In these cards, the contents of only fields 3 to 11 and of columns 79 and 80 need be punched.

The last value of the lens correction must apply to a radial distance which is larger than the largest possible distance.

ii. Cards with measured coordinates

The remaining cards contain the measured coordinates. They are arranged in groups according to the models, and the models are arranged in the sequence in which the triangulation is to be performed.

The first card of a model contains the coordinates of the principal points of the two photographs. Each of the following cards contains the coordinates of corresponding image points in the two photographs. The cards of the points that are to be used for relative orientation come first, and are followed by the cards of any additional points.

The cards are punched as follows:

field 1: Strip-and-model identification (numeric and positive);
field 2: Point identification (numeric and positive);
field 3: x-coordinate in the oriented photograph;
field 4: y-coordinate in this photograph;

field 5: x-coordinate in the new photograph;
field 6: y-coordinate in this photograph;
columns 38-40 of the principal point card: The number of points to be used for relative orientation of the model;
columns 38-40 of the card of a point for relative orientation: If the point is to be used for scaling of the next model and is to be designated as such by a 1-punch: a 1-punch in column 40 of the card of the present model.

As an example, a listing of the input data for two models of a strip flown with a 6'' focal length camera at a height of about 700 m above sea level over terrain with an average elevation of about 300 m is shown in Table 6.

Table 6. Example of input

4		152740	100000	99930	88000	400	87	1
51	30	0	- 130	- 250	- 350	- 450	- 540	- 620
		- 850	- 910	- 970	-1020	-1060	-1090	-1110
		-1090	-1050	-1010	- 960	- 890	- 820	- 740
		- 470	- 370	- 260	- 150	- 20	110	240
		610	720	820	890	940	930	890
		650	550	430	300	150	0	
5070	0000	120343	118614	119715	118943	10		
5070	1001	120523	223974	36397	223122			
5070	1002	132629	122445	47553	121117			
5070	1003	93605	124372	10981	123073			
5070	1004	100176	14086	22127	17638			
5070	1005	200915	222512	116082	222691			
5070	1006	218270	123049	132377	121827			
5070	1007	171382	127079	84606	125669			
5070	1008	199052	18650	113297	20487			
5070	1009	150244	227139	65455	226770			
5070	1010	152235	17789	69085	20341			
5070	149	100177	14084	22127	17638			
5070	151	119773	66341	34725	66653			
5070	31	198200	105289	109110	104017			
5070	185	139233	129921	52923	128414			
5070	16	91349	156199	7608	154360			
5070	184	211999	186798	126626	186079			
5071	0000	119715	118943	120236	119416	10		
5071	1001	116082	222691	49782	215326			
5071	1002	132377	121827	59147	116384			
5071	1003	84606	125669	12999	123122			
5071	1004	113297	20487	35480	18275			
5071	1005	193339	225947	122988	214860			
5071	1006	221938	132637	145954	121191			
5071	1007	178761	124887	103461	116349			
5071	1008	202243	30954	120437	21106			
5071	1009	149634	220311	81151	211450			
5071	1010	152643	26698	72925	21080			
5071	0031	109110	104017	33171	100400			
5071	43	166152	51144	87120	44227			
5071	183	154512	122135	79341	115189			
5071	184	126626	186079	57328	179343			
5071	32	215611	181117	141274	169573			

SUDBURY	1966	RC8	0
- 700	- 780	RC8	1
-1120	-1110	RC8	2
- 650	- 570	RC8	3
370	490	RC8	4
830	750	RC8	5
		RC8	6

5. Restrictions on the input data

The x-axis must be chosen roughly in the direction of the strip. It is not necessary to place the photographs in the comparator with the lines which connect the fiducial marks parallel to the coordinate axes. However, it goes almost without saying that if a strip triangulation is to be performed the orientation of a photograph must not be changed between the measurements for the two models in which it participates.

The positive direction of the x-axis may be chosen at will, but it must be the same for all photographs of a strip.

The calibrated focal length and the measured coordinates must be expressed in the same unit of length, that is, in practice, in microns.

The numbers in fields 1 of the coordinate cards are used to recognize the cards of all points that belong to the same model. Therefore, these numbers must be the same for all points of the same model and they must be different for different models.

In the output, this "strip-and-model" number is listed together with the elements of the orientation matrix and with the coordinates of the projection centre of the new photograph. Therefore, it should contain the number of that photograph and, if possible, a code number for the strip or the job.

At least six points must be used for the relative orientation of a photograph. However, since points that are used for scaling are selected from among the points used for relative orientation, the scaling patterns 2 and 4 require at least 7 and 8 points, respectively.

There is no upper limit to the number of points that may be used for relative orientation of a photograph, but the strip coordinates and parallaxes of only the first 100 of these points will be printed and punched. Those of any remaining points can be obtained by punching for each a second card with the measured coordinates and placing these cards behind the cards for relative orientation.

6. Output

The program uses the on-line printer as its output device. Card output is optional and is needed only for subsequent strip- and block-adjustment.

Model by model, the following lines are printed.

i. One line for each iteration of the relative orientation. These lines contain:

columns 1 -4: The digit 1 for the initial iteration, 2 for the intermediate iteration, and 3 for the final iteration.

columns 10-79: In 14-digit fields and in floating-point notation, the corrections a_1, a_2, a_3, db_Y, and db_Z.

ii. One line for each rejected scale transfer point with:

columns 1-4: The sequence number of the point in the set of scale transfer points.

iii. Three lines for the orientation matrix of the new photograph. These lines contain:

columns 1-4: Strip-and-model number.

columns 10-51: The elements of the orientation matrix, row by row.

iv. One line for the projection centre of the first photograph, but only for the first model of a strip and for independent models.

v. One line for the projection centre of the new photograph.

vi. One line for each of the measured points. The latter lines contain:

field 1: Strip-and-model number.

field 2: Point number (the projection centre is given the point number zero).

field 3: X-coordinate.

field 4: Y-coordinate.

field 5: Z-coordinate.

field 6: The want of intersection, that is the minimum distance between corresponding rays.

In addition, if card output is specified, the above data for the projection centres and the measured points is punched in cards. In the cards, the six fields cover columns 1-4, 5-9, 10-18, 19-27, 28-36, and 37-45, respectively. The cards can be used directly as input cards for the FORTRAN program for strip- and block-adjustment [11].

As an example, the listing of the output data generated by the input data in Table 6 is shown in Table 7.

Table 7. Example of output

```
   1        -0.0327096034  0.0179926156 -0.0016700507  0.0767361294
   2         0.0000678989  0.0027799451  0.0009854273 -0.0023767904
   3         0.0000015948 -0.0000010700 -0.0000005409  0.0000115503
5070         0.9997844773  0.0003001301  0.0207583426
5070        -0.0009775841  0.9994669297  0.0326328184
5070        -0.0207374829 -0.0326460783  0.9992518153

5070    0    200000    400000    600000
5070    0    288000    406544    601138                     0.0163212603
5070 1001    200183    507345    444372          1           -0.0033980163
5070 1002    212152    403786    448839         -6            0.0000154507
5070 1003    173167    405776    446652         -4
5070 1004    179140    291882    442003         -1
5070 1005    281260    504782    445944         -8
5070 1006    297498    404420    447911         10
5070 1007    250040    408299    450196          1
5070 1008    278156    300734    448328        -10
5070 1009    230293    509953    445246          6
5070 1010    231995    298852    446750         11
5070  149    179141    291882    442006          1
5070  151    199445    349124    451296         -2
5070   31    274622    387229    453576          2
5070  185    218483    411062    450479         -1
5070   16    171050    437523    447433         -9
5070  184    291734    468249    447123         15

   1         0.0110158977  0.0312391395  0.0659782488  0.1121171144
   2        -0.0017823685 -0.0010266709 -0.0005603526 -0.0016480327
   3        -0.0000047872 -0.0000028771 -0.0000210818 -0.0000141855
5071         0.9965921121 -0.0646230940  0.0512641957
5071         0.0634396546  0.9976876574  0.0243874670
5071        -0.0527216489 -0.0210521744  0.9983873165

5071    0    367431    415268    600230                      0.0238870539
5071 1001    281261    504762    445974          5           -0.0109504055
5071 1002    297499    404424    447894         -3           -0.0000164685
5071 1003    250032    408296    450161         -9
5071 1004    278157    300749    448352          9
5071 1005    356365    505389    448190         -8
5071 1006    384739    414949    449528          4
5071 1007    343145    407434    449026          9
5071 1008    366171    314874    451849         -3
5071 1009    314374    501415    446898          3
5071 1010    317948    308588    449683         -7
5071   31    274624    387225    453596        -14
5071   43    331185    333829    449916        -17
5071  183    319133    404738    449977         -8
5071  184    291733    468250    447138         -1
5071   32    376985    461299    450801        -15

  TIME   8.4 SECONDS
```

7. Error detection

The program contains a number of tests to detect certain errors in the data deck. If one of these errors occurs, an error message will be printed, the triangulation of the current strip will be discontinued, and the remaining cards of this strip will be skipped.

The computations will be resumed with the next strip. However, this will be done only if earlier a card with zeros or blanks in field 1 is not met. For this reason, it is advisable to punch a positive non-zero number in field 1 of all the cards with the lens correction table.

The error messages and their possible causes are the following:

i. ERROR x. EXIT AT CARD yy

In this case, an error occurs in one of the cards with general information on the strip triangulation. The number x is a code number that indicates the nature of the error. The number yy is the serial number of the last of these cards that has been read. The code numbers and their meaning are the following.

1: The code number of the pattern for the scaling of models, punched in field 1 of the first card, is not in the range from zero to four.

2: The number of values in the lens correction table, punched in field 1 of the second card, is greater than the allowed maximum of 162.

3: According to the serial number on the card, the cards with general information are not in the correct sequence.

ii. ERROR x. EXIT AT CARD yyyy zzzzz

In this case, an error occurs in one of the cards with photograph coordinates. The number x is a code number that indicates the nature of the error. The numbers yyyy and zzzzz are strip-and-model number and point number, respectively, in the last card that has been read. The code numbers and their meaning are here the following.

4: A card is considered to be a principal point card (either because it is placed directly behind the last of the cards with general information or because the model number in the card is different from that in the preceding coordinate card) but the specified number of points for relative orientation, punched in columns 38-40, is smaller than six.

5: The model on which the computer has been operating contains fewer coordinate cards of measured points than the number of points specified for relative orientation.

6: The number of points for relative orientation, punched in columns 38-40 of the principal point card of the preceding model is smaller than the seven or eight which the selected scaling pattern (2 or 4) requires.

The following errors do not produce an error message:

i. No standard scaling pattern has been selected and also in the preceding model no points have been marked by a 1-punch in column 40.

If this occurs, a new triangulation will be started, beginning with the current model.

Because this omission does not cause an error exit, it can be used to perform triangulations of independent models without repeating the cards with general information and without using a card with a negative non-zero number in field 1 to separate the models. For each model, the unit matrix will be retained as the orientation matrix of the first photograph and the value of the base component b_x in the first data card will be used for scaling.

If a standard scaling pattern has not been specified, the absence of a 1-punch in the last model of a strip can be used in the same way for the consecutive triangulation of different strips.

ii. The program has drawn the wrong conclusion concerning whether the photographs are in positive or in negative position.

This conclusion is based upon the sign of the difference of the x-coordinates of the first orientation point, after shifting the origins to the principal points. When the convergence of the camera axes is very great, for some of the points this sign may differ from the sign in the case of parallel axes. If this is the case for the first orientation point, the model that is computed will be upside down. The correct result can be obtained by placing the card of a suitable point directly behind the principal point card.

iii. An error occurs in one of the coordinates punched in the input cards.

The program can detect such an error only if it occurs in a scale transfer point and affects the height of the point. In that case, that point will still be used for relative orientation but it will not be used for scaling. If the affected point shows only a small want of intersection, the error will not appreciably affect the triangulation.

8. Block diagram of the FORTRAN program

The computation starts at Block A and proceeds from each section to the next one, unless specifically stated otherwise. The number above each section is the statement number of its first FORTRAN statement.

Block A Initialize the triangulation

1000

Read the first data card with assorted data on the strip triangulation and read the card or cards with the lens correction table. Check the card sequence and process the data. Read the card with the coordinates of the principal points of the first model.

1018

Initialize counters for triangulation and assign coordinates to the first projection centre.

Block B Perform the relative orientation

1100

Initialize counters for relative orientation. Check and store the number of points to be used for relative orientation. Zero the normal equations.

1110

Use the read-a-point subroutine in section 2010 to read the cards of the points that are to be used in the first iteration of the relative orientation for the current model and to perform the following operations. i. Reduce the measured coordinates to coordinates with origins in the principal points of the two photographs and multiply them by the correction factors for film shrinkage. ii. Use the first point in the first model to decide whether the photographs are in positive or in negative position. iii. Use equations (6. 1), (6. 3), and (6. 5) of Chapter II to correct the photograph coordinates for lens distortion and, if specified, for refraction and for earth curvature. iv. Divide the corrected photograph coordinates by the calibrated focal length and, if the photographs were measured in positive position, change the signs. Further, store the resulting coordinates in the array PHC. If a point is marked by a 1-punch in column 40, store its point number in the array LIST. Compute the correction equation (IV 7. 2) for the first iteration of the relative orientation and add its contribution to the normal equations (IV 5. 4). Finally, solve the normal equations, print the solution, and compute the matrix $\mathbf{R}$ from its parameters with equation (III 4. 6).

1150

Store the computed values of the base components b_Y and b_Z. Store the matrix $\mathbf{R}$ as the orientation matrix $\mathbf{A}_{i+1}$ of the new photograph. If the absolute values of the five orientation parameters are all smaller than 1/30th of a radian, go to section 1300; otherwise go to section 1200.

1200

Zero the normal equations. For each of the earlier read points, compute the correction equation (IV 3. 7) for relative orientation and add its contribution to the normal equations. Solve the normal equations, print the solution, and compute the matrix $\mathbf{R}$. Premultiply the orientation matrix of the new photograph by the matrix $\mathbf{R}$ and add the computed corrections to the base components.

1300

Use section 1200 to perform the final iteration of the relative orientation. This time, however, read also the remaining points, if any, that are to be used for relative orientation and add their contributions to the normal equations.
Go to section 1400.

Block C Scale the model

1400

For the first model of the strip, make the base component b_X equal to the value specified in the first data card.
For all other models, compute the scale factor. For this purpose, if in the preceding model no point has been marked by a 1-punch in column 40, go to section 1410. If one or more points have been marked in this way, go to section 1430.

1410 and 1430

For each point that has been designated as a scale transfer point, replace the distance from the point in the preceding model to the plane $z = 0$ in the common photograph by a scale factor which is the ratio of the distances in the two models.

1440

Compute the mean of the scale factors and store this as the base component b_X. Eliminate in an iterative procedure one at a time those scale factors which differ from the mean by more than 5 parts in ten thousand. If two scale factors differ too much from the mean by practically the same amount, eliminate the second one.
By way of identification, print for each discarded point the number that gives its position in the set of scale transfer points for the current model.

Block D Complete the absolute orientation

1500

Multiply the base components b_Y and b_Z by the scale factor.
For each model except the first one, compute the final values of the orientation matrix and of the base vector by pre-multiplying the above computed values by the orientation matrix of the first photograph of the model. Print the orientation matrix of the new photograph. For the first model only, print the coordinates of the first projection centre.
Compute and print the coordinates of the new projection centre.
If this is specified on the first data card, punch the coordinates.

Block E Compute strip coordinates

1600

Initialize counters for selecting the scale transfer points to be used in the next model.

1610

Use section 1200 to locate the photograph coordinates of the stored points and use the read-a-point subroutine to read the cards and to compute the photograph coordinates of all other points which belong to the same model.
For each point, use subroutines to compute the position vector of the point with respect to an origin in the projection centre of the first photograph of the model and the want of intersection. Then go to section 1620.
Finally, after reading a card with a different strip-and-model number, go to section 1700.

1620

Compute the strip coordinates of the point that is being processed.
Print and, if that is specified, punch the coordinates and the want of intersection.
If the point is a scale transfer point for the next model, compute and store its distance to the plane z = 0 of the new photograph of the model.
Return to section 1200 and use that section as specified in section 1610.

Block F Prepare for the next model

1700

If the last card read has a negative non-zero number in field 1, return to statement 1000 to start the triangulation of the next strip.
If this card has zeros or blanks in that field, end the computations.
Otherwise, the card is the principal point card of the next model. Set counters and shift data in preparation for the triangulation of that model. If one or more distances for scaling that model are available, go to section 1100. If not, go to section 1018.

9. Symbols in the FORTRAN statements

AL, and AL1 to AL9:	Array for the orientation matrix $\mathbf{A}_i$ of the first photograph of a model, and its elements (column after column).
AR, and AR1 to AR9:	Array for the orientation matrix $\mathbf{A}_{i+1}$ of the second photograph of a model, and its elements.
R, and R1 to R9:	Array for the matrix $\mathbf{R}$ for correction of the relative orientation, and its elements.
PP, and PP1 to PP4:	Array for the coordinates of the principal points, and its elements.
W, and U1, V1, U2, V2:	Array for temporary storage of the photograph coordinates of a point in first and second photograph of a model, and its elements.
ID and PHC:	Arrays for point identification and photograph coordinates of points used for the relative orientation.
CLIST:	Array for the corrections for lens distortion.
T, and T1 to T9:	Array for the correction equations, and its elements. Also used for temporary storage.

S:	Array for the normal equations.
SLIST:	Array for the distances and the scale factors computed for each scale transfer point.
F, CX, CY, BX1, CE, CR:	The quantities in fields 3 to 8 of the first data card.
DELR:	The interval of the radial distances for which the lens correction is listed.
X1, Y1, Z1:	The components of the vector $\mathbf{X}_i$ of the point in the first photograph of a model.
X2, Y2, Z2:	The components of the vector $\mathbf{X}_{i+1}$ of the point in the second photograph.
A1, A2, A3, DBY, DBZ:	The parameters of relative orientation.
BX, BY, BZ:	The three base components.
PX, PY, PZ:	The coordinates of the projection centre of the first photograph of a model.
WANT:	The want of intersection.
IX, IY, IZ:	The strip coordinates of the measured points.
LIST and LYST:	Arrays for the point identification of the scale transfer points.
ID1 and ID2:	Locations for storing the model number.
K, KA, KB, KC, KD, MOD:	Indices which control the exits from the subroutines and the transfers from one part of the main routine to another. In particular, K progresses from 1 to 4 during the computation of each model. MOD is equal to 1 for the first model and is equal to 2 for all following models.
K1:	Counter for the points used for relative orientation.
K2:	Counter for locating the coordinates of those points in the array PHC.
K3:	Index which is used in the determination of positive or negative position of the photographs.
K4:	Index which specifies the number of points to be used for relative orientation.
K4MAX:	Index which states how many of these points can be accommodated in arrays in storage.
K5:	Index which is equal to the smaller one of K4 and K4MAX.
K6, K7, K8:	Locations for temporary storage.
KK:	Index which contains the code for the specified scaling pattern.
KK1, KK2:	Indices which indicate whether any point has been marked for scale transfer by means of a 1-punch.
KKK:	Index which indicates whether the correction equations are to be given different weights.

10. Listing of the FORTRAN statements

```
          C     ANALYTICAL STRIP TRIANGULATION.
          C     N.R.C. PROGRAM OF DECEMBER 1966 - G.H.S.
S.0001          DOUBLE PRECISION R(9),AR(12),AL(9),W(4),PP(4),T(9),
               1 S(20),CLIST(162),SLIST(10),PHC(404)
S.0002          DOUBLE PRECISION R1,R2,R3,R4,R5,R6,R7,R8,R9, AR1,AR2,
               1 AR3,AR4,AR5,AR6,AR7,AR8,AR9, BX,BY,BZ, AL1,AL2,AL3,
               2 AL4,AL5,AL6,AL7,AL8,AL9, U1,V1,U2,V2, PP1,PP2,PP3,
               3 PP4, T1,T2,T3,T4,T5,T6,T7,T8,T9, A1,A2,A3,DBY,DBZ,
               4 X1,Y1,Z1,X2,Y2,Z2,PX,PY,PZ, DELR,WANT, F,CX,CY,BX1,
               5 CE,CR
S.0003          DIMENSION        ID(101),LIST(10),LYST(10)
S.0004          COMMON           R1,R2,R3,R4,R5,R6,R7,R8,R9, AR1,AR2,
               1 AR3,AR4,AR5,AR6,AR7,AR8,AR9, BX,BY,BZ, AL1,AL2,AL3,
               2 AL4,AL5,AL6,AL7,AL8,AL9, U1,V1,U2,V2, PP1,PP2,PP3,
               3 PP4, T1,T2,T3,T4,T5,T6,T7,T8,T9, S
S.0005          EQUIVALENCE (R1,R(1)), (AR1,AR(1)), (AL1,AL(1)),
               1 (U1,W(1)), (PP1,PP(1)), (T1,T(1)), (A1,S(16)),
               2 (A2,S(17)), (A3,S(18)), (DBY,S(19)), (DBZ,S(20))
S.0006        1 FORMAT  (I4,I5,F7.3,2F7.5,F7.0,F7.3,F7.4,I7,20X,I2)
S.0007        2 FORMAT  (I4, F5.1, 9F7.2, 6X, I2)
S.0008        3 FORMAT  (I4, I5, 4F7.3, I3)
S.0009        4 FORMAT  (1H I4, 5X, 5F14.10)
S.0010        5 FORMAT  (I4, I5, 4I9)
S.0011       55 FORMAT  (1H  I4, I5, 4I9)
S.0012        6 FORMAT  (6HOERROR I2, 14H. EXIT AT CARD2I6)
S.0013        9 FORMAT  (1H1)
          C
          C     SELECT CARD READER, CARD PUNCH, AND PRINTER
S.0014          IRCD    = 1
S.0015          IWCD    = 2
S.0016          IPR     = 3
          C     CALCULATE ACTUAL COMPUTATION TIME
S.0017       94 FORMAT  (7HO  TIME F5.1, 8H SECONDS)
S.0018          CALL CLOCK (ITEM1)
          C
          C
          C    BLOCK A   INITIALIZE THE TRIANGULATION
          C
          C     READ CODES, FOCAL LENGTH, ETC
          C
S.0019     1000 READ (IRCD,1) KK, KKK, F, CX, CY, BX1, CE, CR, NOCRD, K7
S.0020          CR      = CR / 1000.
S.0021          CE      =(CE / 12756. + CR) / (F*F)
S.0022          WRITE (IPR,9)
S.0023          KK      = KK + 1
S.0024          K8      = 1
S.0025          IF (KK)    2901, 2901, 1001
S.0026     1001 IF (KK-5) 1010, 1010, 2901
          C
          C     READ CORRECTION TABLE
          C
S.0027     1010 K1      = 1
S.0028     1011 K2      = K1 + 8
S.0029          READ (IRCD,2) K3, T3, (CLIST(I),I=K1,K2), K5
```

```
S.0030           IF (K1-1)  1012, 1012, 1014
S.0031      1012 K6     = K3
S.0032           DELR   = T3
S.0033           K8     = 2
S.0034           IF (K6-162) 1014, 1014, 2902
            C     CHECK CARD SEQUENCE
S.0035      1014 K7     = K7 + 1
S.0036           K8     = 3
S.0037           IF (K5-K7)  2902, 1015, 2902
S.0038      1015 K1     = K1 + 9
S.0039           IF (K6-K2)  1016, 1016, 1011
            C     DIVIDE RADIAL CORRECTION BY INTERVAL
S.0040      1016 DO 1017  I = 1,K6
S.0041      1017 CLIST(I) = CLIST(I) / (DELR * 1000.)
            C
            C     READ FIRST CARD OF FIRST MODEL
            C
S.0042           K4MAX  = 100
S.0043           READ (IRCD,3) ID1, ID2, (PP(I),I=1,4), K6
S.0044      1018 K4      = K6
S.0045           MOD    = 1
S.0046           K3     = 1
S.0047           PX     = 200000.
S.0048           PY     = 400000.
S.0049           PZ     = 600000.
            C
            C
            C    BLOCK B   PERFORM THE RELATIVE ORIENTATION
            C
            C     FIRST ITERATION
            C
S.0050      1100 K      = 1
S.0051           KA     = 1
S.0052           K2     = 1
S.0053           KK1    = 1
S.0054           WRITE (IPR,55)
S.0055           DO 1101 I=1,10
S.0056      1101 LIST(I) = -1
S.0057           LI     = 1
S.0058           K5     = K4MAX
S.0059           IF (K4 - K4MAX) 1102, 2030, 2030
S.0060      1102 K5     = K4
S.0061           K8     = 4
S.0062           IF (K4-6) 2903, 2030, 2030
            C
            C     READ FIRST GROUP OF ORIENTATION POINTS
            C
S.0063      1110 DO 1121  K1 = 1,K5
S.0064           GO TO 2010
S.0065      1111 DO 1112  I = 1,4
S.0066           PHC(K2) = W(I)
S.0067      1112 K2     = K2 + 1
            C     TAG POINT FOR SCALING NEXT MODEL
S.0068           IF (LI-10)  1113, 1113, 1120
S.0069      1113 IF (K6 - 1) 1120, 1114, 1120
S.0070      1114 LIST(LI) = ID(K1)
S.0071           LI     = LI + 1
S.0072           KK1    = 2
```

```
               C      CORRECTION EQUATION FOR FIRST ITERATION
               C      T2 IS THE COEFFICIENT OF 2A2+DBZ
               C      T6 IS IN SECOND PART OF EQUATION
S.0073          1120 T1      = V1 * V2 + 1.
S.0074               T2      =-V1 * U2
S.0075               T3      =-U2
S.0076               T4      = U2 - U1
S.0077               T5      = U1 * V2
S.0078               T6      = V2 - V1
               C      FOR THE CONTRIBUTION TO THE NORMAL EQUATIONS,
S.0079               GO TO 2090
S.0080          1121 CONTINUE
               C
               C      TO SOLVE THE NORMAL EQUATIONS,
S.0081               GO TO 2100
               C
S.0082          1150 BY      = DBY
S.0083               BZ      = DBZ
S.0084               DO 1151  I=1,9
S.0085          1151 AR(I)   = R(I)
               C      TEST SIZE OF CORRECTIONS
S.0086               DO 1152  I=16,20
S.0087               IF (30. * S(I)) 1200, 1152, 1200
S.0088          1152 CONTINUE
S.0089               GO TO 1300
               C
               C      SECOND ITERATION, USED ONLY IF
               C      FIRST CORRECTIONS ARE LARGE
               C
S.0090          1200 K       = 2
S.0091          1201 K2      = 1
S.0092               GO TO 2030
               C      USE FIRST GROUP OF ORIENTATION POINTS
S.0093          1202 DO 1204  K1 = 1,K5
S.0094               DO 1203  I=1,4
S.0095               W(I)    = PHC(K2)
S.0096          1203 K2      = K2 + 1
S.0097               GO TO 2040
S.0098          1204 CONTINUE
S.0099               GO TO (2999, 2100, 1302, 1602), K
               C
               C      FINAL ITERATION
               C
S.0100          1300 K       = 3
S.0101               KB      = 1
S.0102               GO TO 1201
S.0103          1301 GO TO (1204, 1303), KB
               C      USE REMAINING ORIENTATION POINTS, IF ANY
S.0104          1302 KB      = 2
S.0105               K1      = K4MAX + 1
S.0106               K8      = K5
S.0107          1303 K8      = K8 + 1
S.0108               IF (K8-K4) 2010, 2010, 2100
               C
               C
               C    BLOCK C   SCALE THE MODEL
               C
S.0109          1400 KA      = 2
```

```
S.0110          GO TO (1460, 1401), MOD
S.0111     1401 KD      = 1
S.0112          BX      = 1.
S.0113          J       = 0
         C      TO COMPUTE SCALE FACTORS,
S.0114          GO TO (1410, 1430), KK2
         C
         C      USE SPECIFIED REGULAR POINT SEQUENCE
         C
S.0115     1410 KKB     = 3
S.0116          GO TO (2999, 1411, 1412, 1414, 1413), KK
S.0117     1411 KKB     = 2
S.0118     1412 KKA     = 2
S.0119          GO TO 1420
S.0120     1413 KKB     = 4
S.0121     1414 KKA     = 1
S.0122     1420 DO 1424  M2 = KKA,KKB
         C      REPLACE DISTANCE IN SLIST BY SCALE FACTOR
S.0123     1421 DO 1422  I = 1,4
S.0124          K7      = 4 * M2 - 4 + I
S.0125     1422 W(I)    = PHC(K7)
         C      FOR X2, X1, D, B*X2, LAMBDA1, AND 0,5 LAMBDA3,
S.0126          GO TO 2040
S.0127     1423 J       = J + 1
S.0128          SLIST(J) = SLIST(J) / T6
S.0129          GO TO (1424, 1432), KK2
S.0130     1424 CONTINUE
S.0131          GO TO 1440
         C
         C      SCALE WITH TAGGED POINTS
         C
S.0132     1430 DO 1432  M1 = 1,10
S.0133          IF (LYST(M1))  1440, 1431, 1431
S.0134     1431 DO 1432  M2 = 1,K5
S.0135          IF (ID(M2)-LYST(M1))  1432, 1421, 1432
S.0136     1432 CONTINUE
         C
         C      MEAN THE SCALE FACTORS
         C
S.0137     1440 T1      = 0.
S.0138          BX      = 0.
S.0139          DO 1443  I = 1,J
S.0140          IF (SLIST(I))  1443, 2999, 1442
S.0141     1442 BX      = BX + SLIST(I)
S.0142          T1      = T1 + 1.
S.0143     1443 CONTINUE
S.0144          BX      = BX / T1
         C      DISCARD ANOMALOUS SCALE FACTORS
S.0145          T2      = 0.
S.0146          DO 1455  I = 1,J
S.0147          IF (SLIST(I))  1455, 2999, 1451
S.0148     1451 T3      = SLIST(I) - BX
S.0149          IF (T3)  1452, 1455, 1453
S.0150     1452 T3      = -T3
S.0151     1453 IF (T3 - .9999999999 * T2) 1455, 1454, 1454
S.0152     1454 K7     =  I
S.0153          T2     =  T3
S.0154     1455 CONTINUE
```

```
S.0155          IF (T2 / BX - .0005) 1500, 1500, 1456
S.0156     1456 SLIST(K7) = -SLIST(K7)
S.0157          WRITE (IPR,55) K7
S.0158          GO TO 1440
          C
S.0159     1460 BX      = BX1
          C
          C
          C     BLOCK D   COMPLETE THE ABSOLUTE ORIENTATION
          C
          C      BASE COMPONENTS AND ORIENTATION MATRIX
          C
S.0160     1500 BY      = BX * BY
S.0161          BZ      = BX * BZ
S.0162          GO TO (1510, 1501), MOD
S.0163     1501 DO 1502 I = 1,9
S.0164     1502 R(I)    = AL(I)
S.0165          J       = 12
S.0166          GO TO 2210
          C
          C      PRINT MATRIX AND COORDINATES OF CENTRES
          C
S.0167     1510 DO 1511  I=1,3
S.0168     1511 WRITE (IPR,4) ID1, AR(I), AR(I+3), AR(I+6)
S.0169          WRITE (IPR,55)
S.0170          K7      = 0
S.0171          GO TO (1512, 1514), MOD
S.0172     1512 IX      = PX
S.0173          IY      = PY
S.0174          IZ      = PZ
S.0175          WRITE (IPR,55) ID1, K7, IX, IY, IZ
S.0176          IF (NOCRD)  1514, 1513, 1514
S.0177     1513 WRITE (IWCD,5) ID1, K7, IX, IY, IZ
S.0178     1514 IX      = PX + BX
S.0179          IY      = PY + BY
S.0180          IZ      = PZ + BZ
S.0181          WRITE (IPR,55) ID1, K7, IX, IY, IZ
S.0182          IF (NOCRD)  1520, 1515, 1520
S.0183     1515 WRITE (IWCD,5) ID1, K7, IX, IY, IZ
          C
          C
          C     BLOCK E   COMPUTE STRIP COORDINATES
          C
          C      INITIALIZE TRIANGULATION
          C
S.0184     1520 KD      = 2
          C      SET COUNTERS FOR THE COMPUTATION OF DISTANCES
S.0185          GO TO (1521, 1527), KK1
S.0186     1521 KKB     = 5
S.0187          GO TO (1600, 1526, 1523, 1524, 1525), KK
S.0188     1523 KKB     = 7
S.0189          KKA     = 6
S.0190          GO TO 1527
S.0191     1524 KKB     = 6
S.0192          KKA     = 4
S.0193          GO TO 1527
S.0194     1525 KKB     = 8
S.0195     1526 KKA     = 5
```

```
S.0196      1527 KKC     = 1
           C
           C      TRIANGULATE STORED POINTS
           C
S.0197      1600 K       = 4
S.0198           KB      = 1
S.0199           K2      = 1
S.0200           GO TO 1202
S.0201      1601 GO TO (1204, 1603), KB
           C
           C      TRIANGULATE ADDITIONAL POINTS
           C
S.0202      1602 KB      = 2
S.0203           K1      = K4MAX + 1
S.0204      1603 GO TO 2010
           C
           C      POSITION VECTOR AND WANT OF INTERSECTION
           C
S.0205      1610 IX      = T4 + PX
S.0206           IY      = T5 + PY
S.0207           IZ      = T6 + PZ
           C      ROUND OFF PROPERLY
S.0208           K7      = WANT * DSQRT(T9) + .5
S.0209           IF (WANT)  1611, 1612, 1612
S.0210      1611 K7      = K7 - 1
S.0211      1612 WRITE (IPR,55) ID1, ID(K1), IX, IY, IZ, K7
S.0212           IF (NOCRD)  1614, 1613, 1614
S.0213      1613 WRITE (IWCD,5) ID1, ID(K1), IX, IY, IZ, K7
S.0214      1614 GO TO (1620, 1603), KB
           C
           C      STORE DISTANCE FOR SCALING NEXT MODEL
           C
S.0215      1620 GO TO (1621, 1630), KK1
S.0216      1621 GO TO (1204, 1622, 1622, 1622, 1622), KK
S.0217      1622 IF (KKA-K1)  1623, 1623, 1204
S.0218      1623 IF (K1-KKB)  1624, 1624, 1204
S.0219      1624 SLIST(KKC) = AR7*(T4-BX) + AR8*(T5-BY) + AR9*(T6-BZ)
S.0220           KKC     = KKC + 1
S.0221           GO TO 1204
           C
S.0222      1630 IF (KKC-10)  1631, 1631, 1204
S.0223      1631 IF (LIST(KKC)-ID(K1))  1204, 1624, 1204
           C
           C
           C     BLOCK F   PREPARE FOR NEXT MODEL
           C
S.0224      1700 K8      = 5
S.0225           GO TO (2904, 1701), KA
S.0226      1701 ID1     = ID2
S.0227           IF (ID1) 1000, 2999, 1702
S.0228      1702 DO 1703  I=1,4
S.0229      1703 PP(I)   = W(I)
S.0230           GO TO (1704, 1706), KK1
S.0231      1704 GO TO (1018, 1706, 1705, 1706, 1705), KK
S.0232      1705 K8      = 6
S.0233           IF (KKB - K4)  1706, 1706, 2904
S.0234      1706 K4      = K6
S.0235           DO 1707  I = 1,10
```

```
S.0236     1707 LYST(I) = LIST(I)
S.0237          KK2     = KK1
S.0238          PX      = PX + BX
S.0239          PY      = PY + BY
S.0240          PZ      = PZ + BZ
S.0241          DO 1708  I = 1,9
S.0242     1708 AL(I)   = AR(I)
S.0243          MOD     = 2
S.0244          GO TO 1100
          C
          C
          C     SUBROUTINES
          C
          C     READ A POINT
S.0245     2010 READ (IRCD,3) ID2, ID(K1), (W(I),I=1,4), K6
S.0246          IF (ID2-ID1)  1700, 2011, 1700
S.0247     2011 U1      = (U1 - PP1) * CX
S.0248          V1      = (V1 - PP2) * CY
S.0249          U2      = (U2 - PP3) * CX
S.0250          V2      = (V2 - PP4) * CY
          C
          C     CORRECT PHOTOGRAPH COORDINATES
S.0251          KC      = 1
S.0252          T1      = U1 * U1 + V1 * V1
          C     USE FIRST POINT TO DEFINE POSITION OF PHOTOGRAPH
S.0253          GO TO (2012, 2021), K3
S.0254     2012 K3      = 2
S.0255          IF (U2-U1) 2013, 2013, 2021
          C     FOR ROTATION TO NEGATIVE POSITION
S.0256     2013 F       =-F
S.0257     2021 T2      = DSQRT (T1)
S.0258          J       = T2 / DELR + 1.
S.0259          T3      = J
S.0260          T3      = DELR * T3 - T2
S.0261          T4 = (T3*CLIST(J)+(DELR-T3)*CLIST(J+1))/T2 +CR +CE*T1
S.0262          K7      = KC + 1
S.0263          DO 2022  I = KC,K7
S.0264     2022 W(I)    = (W(I) + W(I) * T4) / F
S.0265          GO TO (2024, 2999, 2025), KC
S.0266     2024 KC      = 3
S.0267          T1      = U2 * U2 + V2 * V2
S.0268          GO TO 2021
S.0269     2025 GO TO (1111, 2999, 2040, 2040), K
          C
          C     ZERO THE NORMAL EQUATIONS
S.0270     2030 DO 2031  M1 = 1,20
S.0271     2031 S(M1)   = 0.
S.0272          GO TO (1110, 1202, 1202), K
          C
          C     VECTOR X2
S.0273     2040 X2      = AR1 * U2 + AR4 * V2 + AR7
S.0274          Y2      = AR2 * U2 + AR5 * V2 + AR8
S.0275          Z2      = AR3 * U2 + AR6 * V2 + AR9
S.0276          GO TO (2050, 2230),KA
          C
          C     CORRECTION EQUATION FOR SECOND ITERATION
          C     CROSS PRODUCT B * U1
S.0277     2050 T7      = BY - V1 * BZ
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S.0278           T8      = U1 * BZ - 1.
S.0279           T9      = V1 - U1 * BY
           C     CROSS PRODUCT X2 * (B * U1)
S.0280           T1      = Y2 * T9 - Z2 * T8
S.0281           T2      = Z2 * T7 - X2 * T9
S.0282           T3      = X2 * T8 - Y2 * T7
           C     CROSS PRODUCT U1 * X2 AND U1 * X2 . B
S.0283           T4      = X2 - U1 * Z2
S.0284           T5      = U1 * Y2 - V1 * X2
S.0285           T6      = Y2 - V1 * Z2 - T4 * BY - T5 * BZ
S.0286           GO TO (2999, 2090, 2051), K
           C     APPLY WEIGHT
S.0287       2051 IF (KKK)  2090, 2090, 2052
S.0288       2052 T7 = 1./((.14+U1*U1+V1*V1)**2+(.14+U2*U2+V2*V2)**2)
S.0289           T7      = DSQRT(T7)
S.0290           DO 2053  I = 1,6
S.0291       2053 T(I)    = T7 * T(I)
           C
           C     FORM THE NORMAL EQUATIONS
S.0292       2090 M3      = 1
S.0293           DO 2091  M1 =1,5
S.0294           DO 2091  M2 = M1,6
S.0295           S(M3)   = S(M3) + T(M1) * T(M2)
S.0296       2091 M3      = M3+1
S.0297           GO TO (1121, 1204, 1301), K
           C
           C     SOLVE THE NORMAL EQUATIONS
           C     ELIMINATION
S.0298       2100 K7      = 0
S.0299           DO 101   L1 = 2,5
S.0300           M1      = K7 + 1
S.0301           K7      = M1 + 7 - L1
S.0302           M3      = K7
S.0303           M4      = M1 + 1
S.0304           M5      = K7 - 1
S.0305           DO 102  L2 = M4,M5
S.0306           T1      = S(L2) / S(M1)
S.0307           DO 103   M2 = L2,K7
S.0308           M3      = M3 + 1
S.0309        103 S(M3)   = S(M3) - T1 * S(M2)
S.0310        102 S(L2)   = T1
S.0311        101 S(K7)   = S(K7) / S(M1)
S.0312           DBZ     = DBZ / S(19)
           C     BACK SUBSTITUTION
S.0313           M1      = 20
S.0314           DO 105   L1 = 2,5
S.0315           M1      = M1 - 1
S.0316           M2      = 20
S.0317           M3      = M3 - 2
S.0318           S(M1)   = S(M3)
S.0319           DO 105   L2 = 2,L1
S.0320           M3      = M3 - 1
S.0321           S(M1)   = S(M1) - S(M2) * S(M3)
S.0322        105 M2      = M2 - 1
S.0323           GO TO ( 120, 2200, 2200), K
S.0324        120 A2      = A2 - DBZ
           C
           C     ORTHOGONAL MATRIX, COLUMNWISE
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S.0325      2200 WRITE (IPR,4) K, (S(I),I=16,20)
S.0326           T1      = .5 * A1
S.0327           T2      = .5 * A2
S.0328           T3      = .5 * A3
S.0329           R1      = T2 * A2 + T3 * A3
S.0330           R5      = T3 * A3 + T1 * A1
S.0331           R9      = T1 * A1 + T2 * A2
S.0332           R2      = T1 * A2 + A3
S.0333           R4      = T1 * A2 - A3
S.0334           R3      = T3 * A1 - A2
S.0335           R7      = T3 * A1 + A2
S.0336           R6      = T2 * A3 + A1
S.0337           R8      = T2 * A3 - A1
S.0338           T8      = 1. + .5 * R9 + T3 * T3
S.0339           DO 2201  I=1,9
S.0340      2201 R(I)    = R(I) / T8
S.0341           R1      = 1. - R1
S.0342           R5      = 1. - R5
S.0343           R9      = 1. - R9
            C    FOR ACCURATE LAST DIGIT, COMPUTE INSTEAD
            C    T7      = .5 * (R9 + T3 * A3)
            C    T8      = 1. / (1. + T7)
            C    T8      = T8 + ((.5 - T8 + .5) - T7 * T8) / (1. + T7)
            C    DO 2201  I=1,9
            C2201 R(I)   = R(I) * T8
            C    R1      = .5 - R1 + .5
            C    R5      = .5 - R5 + .5
            C    R9      = .5 - R9 + .5
S.0344           J       = 9
S.0345           GO TO (1150, 2210, 2210), K
            C
            C    REPLACE MATRIX AR() BY MATRIX PRODUCT R() * AR()
S.0346      2210 DO 2211  I = 1,J,3
S.0347           T1      = AR(I)
S.0348           T2      = AR(I+1)
S.0349           T3      = AR(I+2)
S.0350           AR(I)   = R1 * T1 + R4 * T2 + R7 * T3
S.0351           AR(I+1)= R2 * T1 + R5 * T2 + R8 * T3
S.0352      2211 AR(I+2)= R3 * T1 + R6 * T2 + R9 * T3
S.0353           IF (J-9)  2999, 2220, 1510
            C
S.0354      2220 BY      = BY + DBY
S.0355           BZ      = BZ + DBZ
S.0356           GO TO (1300, 1300, 1400), K
            C
            C    POSITION VECTOR IN STRIP COORDINATE SYSTEM
            C    X = P1 + LAMBDA1 X1 + 0.5 LAMBDA3 D
            C    VECTOR X1
S.0357      2230 GO TO (2231, 2232), MOD
S.0358      2231 X1      =  U1
S.0359           Y1      =  V1
S.0360           Z1      =  1.
S.0361           GO TO 2234
S.0362      2232 GO TO (2231, 2233), KD
S.0363      2233 X1      = AL1 * U1 + AL4 * V1 + AL7
S.0364           Y1      = AL2 * U1 + AL5 * V1 + AL8
S.0365           Z1      = AL3 * U1 + AL6 * V1 + AL9
            C    CROSS PRODUCT D = X1 * X2, AND D.D
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 2234 T1      = Y1 * Z2 - Z1 * Y2
      T2      = Z1 * X2 - X1 * Z2
      T3      = X1 * Y2 - Y1 * X2
      T9      = T1 * T1 + T2 * T2 + T3 * T3
C     CROSS PRODUCT B * X2
      T4      = BY * Z2 - BZ * Y2
      T5      = BZ * X2 - BX * Z2
      T6      = BX * Y2 - BY * X2
C     LAMBDA1, LAMBDA3, AND 0.5 LAMBDA3
      T7      = (T4 * T1 + T5 * T2 + T6 * T3) / T9
      WANT    = (BX * T1 + BY * T2 + BZ * T3) / T9
      T8      = 0.5 * WANT
C     POSITION VECTOR, REFERRED TO ORIGIN IN FIRST CENTRE
      T4      = T7 * X1 + T8 * T1
      T5      = T7 * Y1 + T8 * T2
      T6      = T7 * Z1 + T8 * T3
      GO TO (1423, 1610), KD
C
C     ERROR MESSAGES
 2901 K5      = K7
 2902 WRITE (IPR,6) K8, K5
C
 2910 READ (IRCD,3) ID2
 2911 IF (ID2) 1000, 2999, 2910
C
 2903 WRITE (IPR,6) K8, ID1, ID2
      GO TO 2910
 2904 WRITE (IPR,6) K8, ID2, ID(K1)
      GO TO 2911
C
 2999 CALL CLOCK (ITEM2)
      SEC     = (ITEM1 - ITEM2) / 76800.
      WRITE (IPR,94) SEC
      CALL EXIT
C
      END
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Notes on the use of the FORTRAN program for analytical strip triangulation in publication AP-PR 34

The use of this program with a compiler that has not exactly the same features as the one used on the IBM S/360 at the NRC laboratories may require one or more of the following modifications in the program.

i. Appropriate changes in the read and write statements and in the associated statements S.0014, 15, and 16 may have to be made.

ii. The statements S.0017, 18, 388, 389, and 390 which make use of the CLOCK subroutine in the compiler package may have to be eliminated. In that case, statement S.0391 receives the label 2999. Replace S.0391 by the appropriate STOP statement.

iii. If the compiler requires listing the variables in COMMON from the end of the COMMON storage to the beginning, as for the IBM 1620, the sequence of the variables in statement S.0004 must be reversed.

iv. If the compiler cannot deal with some of the entries into DO-loops, as in the case of the compilers for the IBM 7094, some or all of the possibly offending DO-loops must be changed into IF-loops. Insert the following statements in the place of the ones in the program listing:

```
S.0063    1110 K1 = 1

S.0080    1121 K1 = K1 + 1
S.0081         IF (K1-K5) 2010, 2010, 2100

S.0093    1202 K1 = 0
          1205 K1 = K1 + 1

S.0098    1204 IF (K1-K5) 1205, 1206, 1206
S.0099    1206 GO TO (2999, 2100, 1302, 1602), K

S.0122    1420 M2 = KKA

S.0130    1424 M2 = M2 + 1
S.0131         IF (M2-KKB) 1421, 1421, 1440

S.0132    1430 M1 = 0
          1435 M1 = M1 + 1

S.0134    1431 M2 = 0
          1436 M2 = M2 + 1

S.0136    1432 IF (M2-K5) 1436, 1437, 1437
          1437 IF (M1-10) 1435, 1440, 1440
```

Prospective users are requested to punch their own card deck from the listing. The listing was typed in order to make it easy to read and it was proven to be free from errors by punching a card deck from the listing and testing this deck. In case difficulties should occur in making a program operational, some assistance can be obtained from the author.

Ottawa, March 1967