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ON THE THEORY OF GUIDE-FED LINEAR ARRAYS

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ON THE THEORY OF GUIDE-FED LINEAR ARRAYS

The proper understanding of the feeding of linear arrays is of paramount importance for the effective design of arrays to provide scanning beams. In the course of the session devoted to arrays at the recent Antenna Conference held at M.I.T., some statements were made which seemed designed more to support the procedures which have been used than to throw light on the functioning of the feed in those arrays. The idea of coupling coefficient continues in use and it is thought by some to be a satisfactory concept for handling long arrays of "weakly-coupled" elements. It is difficult to see to what extent the coupling can be said to be weak when the shunt impedances of the loading elements on the guide are of the order of the guide impedance and therefore draw a small fraction of the power in the feed only because the array of elements tends to force minima in the standing wave systems to the points of loading. In the second place, the guide with half-wave spaced elements is popularly regarded as a separate type of feed distinct from feeds with other spacing and so the only one for which one must treat of impedance. It is the intention of the present report to point out that these hypotheses and this classification are erroneous and to outline more fundamental conceptions of the behaviour of linear arrays. Impedance is a two-parameter quantity. The dipole arrays usually have had only one adjustment -- length of probe. How then can one expect to have produced a prescribed amplitude and phase distribution in the excitation of the elements of the array?

1. In what follows, we shall assume shunt-loading but the principle of the method is applicable to any type of loading whatever.

Let γ_r = shunt admittance of the rth load,

V_t = voltage in the equivalent transmission line at this point

Suppose the loads are spaced equidistance d apart, or θ radians reckoned on the basis of the unloaded line. The difference equation satisfied by the voltage is

$$V_{r+1} - 2\cos\theta\, V_r + V_{r-1} = -j\gamma_{r-1}\sin\theta\, V_r \; . \; . \; . \; . \; . \; (1)$$

Now in a long array the γ_r should be small for weak coupling in the proper physical sense of negligible local disturbance of the feed, and γ_r is not a rapidly changing function of r. Let us proceed to treat (1) by the analogue of the Method of Variation of Parameters.

Let

$$V_r = A_r\omega^r + B_r\omega^{-r} \text{ where } \omega = e^{j\theta} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

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Since we have introduced two variables A_r and B_r in place of V_r we are at liberty to impose another condition on them: we choose

$$(A_{r+1} - A_r)\,\omega^{r+1} + (B_{r+1} - B_r)\,\omega^{-(r+1)} = 0 \quad \ldots\ldots\ldots\ldots (3)$$

so that

$$V_{r+1} - V_r = A_r\,(\omega^{r+1} - \omega^{r}) + B_r\,(\omega^{-(r+1)} - \omega^{-r}) \quad \ldots\ldots\ldots (4)$$

Hence (1) becomes

$$V_{r+1} - 2\cos\theta\, V_r + V_{r-1} = (V_{r+1} - V_r) - (V_r - V_{r-1}) + (2 - \omega - \omega^{-1})V_r$$

$$-j\sin\theta\,\gamma_{r+1}(A_r\omega^r + B_r\omega^{-r}) = (A_r - A_{r-1})\,(\omega^r - \omega^{r-1}) + (B_r - B_{r-1})$$

$$(\omega^{-r} - \omega^{-r+1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

In consequence of the smallness of γ_r we proceed on the assumption that A_r and B_r do not vary rapidly with r and hence we may replace (3) and (5) by differential equations in which the index r is replaced by the variable x/d.

Thus

$$\frac{dA}{dx}\,\omega^{x/d} + \frac{dB}{dx}\,\omega^{-x/d} = 0 \quad \ldots\ldots\ldots\ldots\ldots (3')$$

and

$$\frac{dA}{dx}\,\omega^{x/d}\,(1 - \omega^{-1}) + \frac{dB}{dx}\,\omega^{-x/d}\,(1 - \omega) = -\,\frac{j\sin\theta\,\gamma\left(\frac{x}{d}\right)}{d}\,(A\omega^{x/d} + B\omega^{-\frac{x}{d}}) \quad \ldots\ldots\ldots\ldots (5')$$

or

$$\left\{\frac{dA}{dx}(1 - \omega^{-1}) + \frac{A\gamma(\omega - \omega^{-1})}{2d}\right\}\omega^{x/d} + \left\{\frac{dB}{dx}(1 - \omega) + \frac{B\gamma(\omega - \omega^{-1})}{2d}\right\}\omega^{-\frac{x}{d}} = 0 \quad \ldots\ldots\ldots\ldots (5'')$$

Comparison of (3') and (5'') shows that

$$\left.\begin{aligned} \frac{dA}{dx}(1 - \omega^{-1}) + \frac{A\gamma(\omega - \omega^{-1})}{2\,d} &= \lambda\,\frac{dA}{dx} \\ \frac{dB}{dx}(1 - \omega) + \frac{B\gamma(\omega - \omega^{-1})}{2d} &= \lambda\,\frac{dB}{dx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6)$$

where λ is a multiplier still to be determined.

$$\text{If } f(x) = \int_0^x \gamma \, dx \quad \text{and } f(L) = \Gamma$$

the solution of (6) may be written

$$A = A_o \exp\left\{ - \frac{(\omega - \omega^{-1})}{2d(1 - \lambda - \omega^{-1})} \; f(x) \right\}$$

$$B = B_L \exp\left\{ - \frac{(\omega - \omega^{-1})}{2d(1 - \lambda - \omega)} \; \left[f(x) - \Gamma \right] \right\} \quad \ldots \ldots \ldots (7)$$

The impedance at any point of the line, an integral number of half-wave-lengths from the origin, is determined by

$$Z = \frac{A + B}{A - B}$$

$$\text{i.e.} \quad \frac{A}{B} = \frac{Z + 1}{Z - 1} = \frac{1}{W} \quad \ldots \ldots \ldots (8)$$

If W_i is the value of W corresponding to the input impedance Z_i, which can always be calculated given the loads, we have

$$\frac{A_o}{B_L} = \frac{1}{W_i} \exp\left\{ \frac{\omega - \omega^{-1}}{(1 - \lambda - \omega)} \; \frac{\Gamma}{2d} \right\} \quad \ldots \ldots \ldots (9)$$

Correspondingly for the terminating impedance

$$\frac{1}{W_T} = \frac{A_o}{B_L} \exp\left\{ - \frac{(\omega - \omega^{-1})}{(1 - \lambda - \omega^{-1})} \; \frac{\Gamma}{2d} \right\} \quad \ldots \ldots \ldots (10)$$

Now eliminate A_o/B_L between (9) and (10) and we obtain a quadratic equation to determine λ in terms of the input and terminating impedances assumed.

The relation is

$$(1 - \lambda)^2 - 2 \cos \theta \, (1 - \lambda) + 1 = - p \sin^2 \theta$$

$$\text{where } p = \frac{2\Gamma}{d \log_e \left(\frac{W_i}{W_T} \right)} \quad \ldots \ldots \ldots (11)$$

$$\text{Thus } 1 - \lambda = \cos \theta \pm j \sin \theta \sqrt{p + 1} \quad \ldots \ldots \ldots (12)$$

In equation (12) the + sign corresponds to the propagation of energy in the direction of x increasing.

We now consider the following cases:

(i) $|p| \ll 1$.

We have from (12)

$$1 - \lambda = \omega + \frac{j\, p \sin \theta}{2} \text{ approximately}$$

and

$$A = A_o \exp\left\{-\frac{1}{2d} f(x)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$B = B_L \exp\left\{-\frac{2}{pd}\left\{f(x) - \Gamma\right\}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

At the input end:

$$B = B_L\, e^{\frac{2\Gamma}{pd}}$$

Hence reflected waves with the corresponding mismatch can be kept down by making Γ small and $W_1 \approx W_T$. That is, we should have real weak coupling and consequently insufficient abstraction of power from the feed.

(ii) $p = -1$:- $1 - \lambda = \cos \theta$.

and

$$\left.\begin{aligned} A &= A_o \exp\left\{-\frac{f(x)}{d}\right\} \\ B &= B_L \exp\left\{\frac{f(x) - \Gamma}{d}\right\} \end{aligned}\right. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

$$\frac{W_1}{W_T} = e^{\frac{-2\Gamma}{d}}$$

If the guide is terminated with a suitable impedance this condition might be fulfilled. Indeed, values of p in the vicinity of -1 may be expected to give behaviour not greatly different from (14) in which the striking characteristic is the extinction of the reflected wave towards the input. There is in this respect here a strong resemblance with the feed in a properly loaded line with elements half-wave apart.

(iii) <u>Half-wave spacing</u>: We return to equations (7) and approximate to the exponentials

$$A = A_0\left(1 - \frac{\omega - \omega^{-1}}{2d\,(1 - \lambda - \omega^{-1})}\, f(x)\right)$$

$$B = B_L\left(1 - \frac{\omega - \omega^{-1}}{2d\,(1 - \lambda - \omega)}\left[f(x) - \Gamma\right]\right)$$

and from equation (ii) $1 - \lambda = \cos\theta \quad (p = -1)$

$$A = A_0\left(1 - \frac{f(x)}{d}\right)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$
$$B = B_L\left(1 + \frac{f(x) - \Gamma}{d}\right)$$

These equations should be compared with the exact equations

$V_r = (-1)^r\, V_0$ where V_0 is a constant

and

$$r = (-1)^r\left[I_0 - V_0 \sum_{s=1}^{r} V_s\right] \ldots\ldots\ldots\ldots\ldots (16)$$

There is no attenuation of the voltage wave, but the current wave falls off in accordance with (16). If the loads had been series ones the current wave would not attenuate but the voltage wave would.

Equation (15) shows that in order that the input be close to a match

$$\frac{\Gamma}{d} \sim 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Now $\frac{\Gamma}{d} = \sum_{r=1}^{N} \gamma_r$ where N = number of elements, thus our condition (17) is equivalent to the statement that $\bar{\gamma}$, the average value of γ, is about $\frac{1}{N}$.

It is possible to terminate the guide with a reflecting plunger, thus making Z_T a large reactance and $W_T \rightarrow +1$. By the proper choice of the γ_r a match can be obtained in the input; roughly speaking $\bar{\gamma} = \frac{1}{N}$. In our arrays this condition is met and the array tolerates change in θ, due to change in the frequency used up to ±1/2% for N about 50. The beamsplit which takes place on further change of θ is due to phase reversals which occur at points in the feed determined by the difference between the period d of spacing and the half-wave length in the guide.

It appears, therefore, that in feeding a long array off half-wave spacing the main problem is not that of obtaining a match (which ought to be no problem at all at half-wave spacing) but rather to secure a good approximation to the desired phase distribution along the array, without reversals. This may be secured by introducing the proper distribution of susceptance among the loads.

II. The method of Section I was applied on the assumption that γ_r is small but its application is not limited to weak loading provided that the dependence of γ_r on r is not rapid. For example let

$$\gamma_r = \gamma_o + g_r$$

The difference equation may be written

$$V_{r+1} - (2 \cos \theta - j \gamma_o \sin \theta) V_r + V_{r-1} = - j \sin \theta \, g_{r-1} V_r \quad \ldots (18)$$

Let x_1 and x_2 be the roots of the equation

$$x^2 - (2 \cos \theta - j \gamma_o \sin \theta) x + 1 = 0 \quad \ldots (19)$$

and

$$V_r = A_r x_1^r + B_r x_2^r \quad \ldots (20)$$

Now apply the method of variation of parameters, approximate by differential equations and use the input and terminating impedance data.

III. For the general type of loading, the difference equation may be set up from the equations which yield the transformation of voltage and current on the line from one loading point to the antecedent one: thus let

$$V_{r-1} = h_r V_r + k_r i_r \quad \ldots (21)$$

$$i_{r-1} = m_r V_r + n_r i_r \quad \ldots (21)$$

then

$$k_r \left(m_{r+1} - \frac{n_{r+1} h_{r+1}}{k_{r+1}}\right) V_{r+1} + \left(h_r + \frac{k_r n_{r+1}}{k_{r+1}}\right) V_r - V_{r-1} = 0 \quad \ldots (22)$$

As an example consider a π-section load in which γ is the series impedance in the equivalent T and α the shunt admittance in the Π.

$$V_{r-1} = \frac{(1 + \alpha\gamma) \cos \theta + 2j\alpha \sin \theta}{1 - \alpha\gamma} V_r + \frac{2\gamma \cos \theta + j \sin \theta (1 + \alpha\gamma)}{1 - \alpha\gamma} i_r$$

$$i_{r-1} = \frac{2\alpha \cos \theta + j \sin \theta (1 + \alpha\gamma)}{1 - \alpha\gamma} V_r + \frac{(1 + \alpha\gamma) \cos \theta + 2j \gamma \sin \theta}{1 - \alpha\gamma} i_r \quad \ldots (23)$$

assuming that the loads are identical.

$$V_r = A x_1^r + B x_2^r$$

where x_1 and x_2 are the roots of

$$\left[(1 + \alpha\gamma)^2 - 4\alpha\gamma\right] x^2 - 2(1 - \alpha\gamma) \left[(1 + \gamma\alpha) \cos \theta + j(\alpha + \gamma) \sin \theta\right] x + (1 - \alpha\gamma)^2 = 0 \quad \ldots (24)$$

When $\alpha = \gamma$ the π-section load presents a match if terminated by a match.

For a wave in the direction of θ increasing

$$x = \frac{1-\alpha}{1+\alpha} \cdot e^{-j\theta} \quad \ldots\ldots\ldots\ldots \quad (25)$$

Thus the fraction of power extracted by the load is

$$1 - \left| \frac{1-\alpha}{1+\alpha} \right|^2 \quad \ldots\ldots\ldots\ldots \quad (26)$$

and the electrical spacing between elements in the loaded guide is effectively

$$\theta - \arg \frac{1-\alpha}{1+\alpha} \quad \ldots\ldots\ldots\ldots \quad (27)$$

on account of the phase change introduced by the general π-section.

The use of such loads seems to commend itself as being the most satisfactory way of removing the matching problem for the feed. It allows a predictable amplitude and phase distribution among the radiators, it gives freedom in choice of the spacing of the elements, and allows beam-swing by phase change in the feed.

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