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THE EFFECTS OF THE DYNAMIC CHARACTERISTICS OF A VIBRATOR ON THE FORCED VERTICAL VIBRATIONS OF PILES

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PREFACE

Pile foundations are used extensively on building sites which contain soils with inadequate strength or undesirable deformational properties. Such foundations often represent a considerable proportion of the total construction costs; improvements in installation methods might therefore result in considerable savings.

The Division of Building Research has maintained an interest in the performance as well as in the method of installation of pile foundations. Exploratory measurements carried out by members of the Division on the vibratory effects of pile driving have established the need for a more realistic theoretical treatment of this complex phenomenon. This translation is made available in the hope that the present work may provide a basis for a better understanding of the pile driving phenomenon and permit a more rational quantitative treatment of the subject.

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N.B. Hutcheon
Director

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TECHNICAL TRANSLATION 1502

Title: The effects of the dynamic characteristics of a vibrator on the forced vertical vibrations of piles

(Zavisimost' amplitudy vynuzhdennykh vertikal'nykh kolebanii svai i shpunta ot parametrov vibratora)

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THE EFFECTS OF THE DYNAMIC CHARACTERISTICS OF A VIBRATOR ON THE FORCED VERTICAL VIBRATIONS OF PILES

1. Introduction

The paper examines the relation between the vibratory characteristics of piles driven into soil by vibration and the characteristics of the vibrator, i.e., the driving moment and number of revolutions.

The first experimental study of this relation was carried out by D. D. Barkan, V. P. Sukharev and V. N. Tupikov(1) in 1951. They found that the linear vibration theory, according to which the bearing capacity of soil is assumed to be proportional to the displacement, is not always in agreement with experimental data on the relation between the amplitude and frequency.

The resonance curves obtained in these experiments referred to a small range of rpm. The curves which extrapolate experimental data beyond this range are somewhat arbitrary, especially if we consider that the frequency range in the experiments was an unstable zone.

Owing to the fact that individual experiments in this field invariably produce some erroneous data, generalizations can be made only on the basis of a large number of experiments.

Our tests involved the sinking of 170 piles (12, 15, 20, 25 and 31 cm in diameter) and 2 sheet piles (ShP and ShK) about 4 m in length. The cross-sectional areas of sheet piles were 70 and 60 cm^2 respectively. (For description of experimental site see the article by N. A. Preobrazhenskaya in this collection of papers).

The total weight of the vibrator was 700 kg, and the maximum eccentric moment 660 kgcm.

The amplitude of vibrations and the power used by the motor of the vibrator were determined for four values of the driving moments and revolutions up to 1700 rpm.

The amplitude and power changed relatively little with depth, which was evidently due to specific geology and relatively small depth of sinking. Geology and testing techniques are described in the paper by Preobraznenskaya.

2. Theoretical Resonance Curves for Different Hypotheses of the Bearing Capacity of Soil

To find the relation between the amplitude of soil vibration and the characteristics of the vibrator, i.e., the driving moment and rpm, let us examine the simplest case, i.e., the vibration of mass with one degree of freedom. Let us represent the bearing capacity of soil in the form of two components:

$$R = R_1(z) + R_2(\dot{z}),$$

where $R_1(z)$ is the component dependent on the displacement of z;
$R_2(\dot{z})$ is the component dependent on the vibration rate, i.e., the damping force or force of friction.

Let us examine the following cases:

1. $R_1(z)$ is nonlinearly dependent on z, while $R_2(\dot{z})$ is proportional to velocity (viscous friction).

2. Various hypotheses concerning damping forces $R_2(\dot{z})$ during displacement proportional to R(z).

3. Forced vibrations of a pile during separation from the soil.

First case. In a conventional linear theory, $R_1(z)$ is assumed to be proportional the the displacement, which is true for small amplitudes.

However, in the case of a vibrator with a large driving moment, the amplitudes may be fairly large, and therefore the relation of $R_1(z)$ to the displacement must obey a more complex law, e.g.:

$$R_1(z) = kz - dz^3 + ez^5.$$

If $R_2(\dot{z}) = \alpha\dot{z}$ the equation for the vibration is as follows:

$$\frac{Q}{g}\ddot{z} + R(z) + \alpha\dot{z} = \frac{Q_0\varepsilon}{g}\,\omega^2 \sin\omega t\,,$$

where $\frac{Q_0\varepsilon}{g}\,\omega^2$ is the perturbation force of the vibrator; Q is the weight of pile and vibrator, including the motor, etc.

On dividing the equation by $\frac{Q}{g}$ we obtain:

$$\ddot{z} + \lambda^2 z - \gamma z^3 + \delta z^5 + 2n\dot{z} = A_\infty \omega^2 \sin\omega t; \qquad (1)$$

$$A_\infty = \frac{Q_0\varepsilon}{Q}. \qquad (2)$$

Let us take the following equation as a first approximation:

$$z = a\sin\omega t - b\cos\omega t = A\sin(\omega t + \varphi),\ \mathrm{tg}\,\varphi = \frac{-b}{a}, \qquad (3)$$

then

$$\ddot{z} = -\lambda^2 z + \gamma z^3 - \delta z^5 - 2n\dot{z} + A_\infty \omega^2 \sin\omega t. \qquad (4)$$

On substituting equation (3) into the right half of (4) and replacing the squares, cubes and products of trigonometric factors by their expressions in terms of higher harmonics including the fifth, we obtain after integration:

$$\begin{aligned}
\omega^2 z = {}& \sin\omega t\left(a\lambda^2 - \frac{3}{4}\gamma a^3 - \frac{3}{4}\gamma ab^2 + \frac{5}{8}\delta a^5 + \frac{10}{8}a^3b^2 + \right.\\
& \left. + \frac{5}{8}ab^4 + 2n\omega b - A_\infty\omega^2\right) + \cos\omega t\left(-b\lambda^2 + \frac{3}{4}\gamma a^2 b + \right.\\
& \left. + \frac{3}{4}\gamma b^3 - \frac{5}{8}\delta a^4 b - \frac{10}{8}a^2b^3 - \frac{5}{8}b^5 + 2n\omega a\right) + \\
& + \sin 3\omega t\left(\frac{\gamma a^3}{36} - \frac{3\gamma ab^2}{36} - \frac{5}{144}\delta a^5 + \frac{10a^3b^2}{144}\delta + \frac{15}{144}\delta ab^4\right) + \\
& + \cos 3\omega t\left(\frac{-3\gamma a^2 b}{36} + \frac{\gamma b^3}{36} + \frac{15}{144}\delta a^4 b + \frac{10}{144}\delta a^2 b^3 - \frac{5}{144}\delta b^5\right) + \\
& + \frac{\sin 5\omega t}{400}\delta\left(a^5 - 10a^3b^2 + 5ab^4\right) + \frac{\cos 5\omega t}{400}\delta\left(-5a^4b + 10a^2b^3 - b^5\right).
\end{aligned} \qquad (5)$$

Let us define a and b such that the coefficients of sin ωt and cos ωt will be identical for (3) and (6) giving:

$$a = \frac{1}{\omega^2}\left(a\lambda^2 - \frac{3}{4}\gamma a A^2 + \frac{5}{8}\delta a A^4 + 2n\omega b - A_\infty \omega^2\right);$$
$$-b = \frac{1}{\omega^2}\left(-b\lambda^2 + \frac{3}{4}\gamma b A^2 - \frac{5}{8}\delta b A^4 + 2n\omega a\right) \qquad (6)$$

or

$$a(\lambda^2 - \omega^2) - \frac{3}{4}\gamma a A^2 + \frac{5}{8}\delta A^4 + 2n\omega b - A_\infty \omega^2 = 0;$$
$$-b(\lambda^2 - \omega^2) + \frac{3}{4}\gamma b A^2 - \frac{5}{8}\delta b A^4 + 2n\omega a = 0. \qquad (7)$$

From this we find a and b, which we shall identify with subscript 1:

$$a_1 = \frac{A^2}{A_\infty^2 \omega^2}\left[\frac{5}{8}\delta A^4 - \frac{3}{4}\gamma A^2 - (\omega^2 - \lambda^2)\right];$$
$$b_1 = \frac{A^2}{A_\infty^2 \omega^2} 2n\omega; \qquad (8)$$

Then the coefficients at high harmonics for sin and cos will be as follows:

$$a_3 = \frac{a_1}{36\omega^2}\left[\gamma\left(a_1^2 - 3b_1^2\right) + \frac{5}{4}\delta\left(-a_1^4 + 2a_1^2 b_1^2 + 3b_1^4\right)\right];$$
$$b_3 = \frac{-b_1}{36\omega^2}\left[\gamma\left(b_1^2 - 3a_1^2\right) + \frac{5}{4}\delta\left(-b_1^4 + 2a_1^2 b_1^2 + 3a_1^4\right)\right];$$
$$a_5 = \frac{a_1\delta}{400\omega^2}\left(a_1^4 - 10a_1^2 b_1^2 + 5b_1^4\right); \qquad (9)$$
$$b_5 = \frac{-b_1\delta}{400\omega^2}\left(-b_1^4 + 10a_1^2 b_1^2 - 5a_1^4\right);$$
$$A_1 = \sqrt{a_1^2 + b_1^2}; \quad A_3 = \sqrt{a_3^2 + b_3^2} = \frac{1}{36\omega^2} A_1^3\left(\gamma - \frac{5}{4}\delta A_1^2\right);$$
$$A_5 = \sqrt{a_5^2 + b_5^2} = \frac{\delta}{400\omega^2} A_1^5.$$

and consequently the displacement will be

$$z = (a_1 \sin \omega t - b_1 \cos \omega t) + (a_3 \sin 3\omega t - b_3 \cos 3\omega t) + (a_5 \sin 5\omega t - b_5 \cos 5\omega t). \qquad (10)$$

$$a_1^2 + b_1^2 = A_1^2; \qquad (11)$$

From (8) we obtain:

$$A^2 = \frac{A_\infty^2 \omega^4}{4n^2\omega^2 + \left(\omega^2 - \lambda^2 + \frac{3}{4}\gamma A^2 - \frac{5}{8}\delta A^4\right)^2}, \qquad (12)$$

(in the case of A_1 subscript 1 will be omitted).

For the phase shift of the first harmonic we have:

$$\operatorname{tg} \varphi = \frac{-b_1}{a_1} = \frac{2n\omega}{\omega^2 - \lambda^2 + \frac{3}{4}\gamma A^2 - \frac{5}{8}\delta A^4}, \qquad (13)$$

hence

$$\sin \varphi = \frac{b_1}{A} = \frac{2nA}{A_\infty \omega}.$$

Let us assume for the sake of simplicity that

$$y = \frac{3}{4}\gamma - \frac{5}{8}\delta A^2; \quad x = 4n^2. \qquad (14)$$

Then, on solving (12) with respect to ω^2, we obtain

$$\omega^2 = \frac{A^2\left(\lambda^2 - A_y^2 - \frac{x}{2}\right) \pm A\sqrt{A_\infty^2\left(\lambda^2 - A_y^2\right)^2 - xA^2\left(\lambda^2 - A_y^2 - \frac{x}{4}\right)}}{A^2 - A_\infty^2} \qquad (15)$$

Let us proceed to generalized coordinates. Let

$$\xi = \frac{4n^2}{\lambda^2}; \quad a = \frac{A}{A_\infty}; \quad u = \frac{3}{4} \cdot \frac{\gamma A_\infty^2}{\lambda^2};$$

$$v = \frac{5}{8} \cdot \frac{\delta A_\infty^4}{\lambda^2}; \quad \beta = u - a^2 v, \quad \bar{\omega} = \frac{\omega}{\lambda}; \qquad (16)$$

then

$$\bar{\omega}^2 = \frac{1}{a^2-1}\left\{a^2\left(1-a^2\beta-\frac{\xi}{2}\right) \pm a\sqrt{(1-a^2\beta)^2-\xi a^2\left(1-a^2\beta-\frac{\xi}{4}\right)}\right\}. \qquad (17)$$

at $\xi = 0$

$$\bar{\omega}^2 = \frac{a(1-a^2\beta)}{a+1}. \qquad (17')$$

One of the limits of $\bar{\omega}^2$ at $a \to 1$ will evidently tend towards infinity ($\bar{\omega} = \infty$).

The second limit of $\bar{\omega}^2$ at $a \to 1$ will be

$$\lim_{a \to 1} \bar{\omega}^2 = \frac{(1-\beta)^2}{2\left(1-\beta-\frac{\xi}{2}\right)}. \qquad (17'')$$

In abstract coordinates equation (10) will assume the following form:

$$a_1 = -\frac{A_\infty}{\bar{\omega}^2}a^2\left(a^2\beta+\bar{\omega}^2-1\right); \quad b_1 = \frac{A_\infty}{\bar{\omega}^2}a^2 v\bar{\omega};$$

$$a_3 = \frac{a_1}{\bar{\omega}^2}\left[\frac{u}{27A_\infty^2}\left(a_1^2-3b_1^2\right) + \frac{v}{18A_\infty^4}\left(-a_1^4+2a_1^2b_1^2+3b_1^4\right)\right];$$

$$b_3 = \frac{-b_1}{\bar{\omega}^2}\left[\frac{u}{27A_\infty^2}\left(b_1^2-3a_1^2\right) + \frac{v}{18A_\infty^4}\left(-b_1^4+2a_1^2b_1^2+3a_1^4\right)\right];$$

$$a_5 = \frac{a_1 v}{250\bar{\omega}^2A_\infty^4}\left(a_1^4-10a_1^2b_1^2+5b_1^4\right);$$

$$b_5 = \frac{b_1 v}{250\bar{\omega}^2A_\infty^4}\left(b_1^4-10a_1^2b_1^2+5a_1^4\right);$$

$$A = \sqrt{a_1^2+b_1^2}; \quad A_3 = \frac{A_\infty a^3}{36\bar{\omega}^2}\left(\frac{4}{3}u-2va^2\right);$$

$$A_5 = \frac{A_\infty a^5 v}{250\bar{\omega}^2}.$$

To avoid possible incompatibility, let us examine the relation between the bearing capacity of soil and displacement.

The bearing capacity of soil has the following form:

$$R = m\,(\lambda^2 z - \gamma z^3 + \delta z^5)\,.$$

Let

$$\frac{z}{A_\infty} = \zeta$$

then

$$R = \lambda^2 A_\infty m \left(\zeta - \frac{4}{3} u\zeta^3 + \frac{8}{5} v\zeta^5\right);$$
$$R' = \lambda^2 m\,(1 - 4u\zeta^2 + 8v\zeta^4).$$

It is natural to assume that with the increase in displacement (amplitude) the bearing capacity will increase also, i.e., R' > 0.

On solving the equation R' = 0 with respect to ζ^2, we find:

$$\zeta^2 = \frac{u}{4v} \pm \frac{1}{4v}\sqrt{u^2 - 2v}\,.$$

Consequently, at $v > \frac{u^2}{2}$, $R' > 0$.

In the treatment that follows we shall take $v = \frac{u^2}{2}$, then

$$R' = \lambda^2 m\,(1 - 2u\zeta^2)^2,$$

i.e., R' is always positive or equal to zero.

The calculations show that the error due to the neglect of the third harmonic does not exceed 20% of the value of the amplitude for u = 8, and is considerably less for other parameters. The error due to neglect of the fifth harmonic is quite negligible. Therefore, additional harmonics may be ignored.

Figure 1 shows the resonance curves calculated from (17). In every case $v = \frac{u^2}{2}$.

If we use the first harmonic only, then the instantaneous power will be equal to:

$$W_{inst} = \frac{Q_0 \varepsilon \omega^2}{g}\,\frac{dz}{dt} = \frac{Q_0 \varepsilon}{2g} A\omega^3\,[\sin(2\omega t + \varphi) - \sin\varphi].$$

and consequently, the power for the cycle of vibrations $T = \frac{2\pi}{\omega}$ will be equal to:

$$W = \frac{1}{T}\int_0^T W_{inst}\, dt = \frac{A^2\omega^2 nQ}{g}. \qquad (18)$$

Figure 2 illustrates the relation between power and frequency at different values of the attenuation coefficient for two values of u:

u = 0(v = 0) and u = 0.5(v = 0.125).

<u>Second case</u>. Let us now examine various hypotheses concerning the friction forces.

The general form of the vibration equation for any relation between damping forces and velocity $f(\dot{z})$ is as follows:

$$m\ddot{z} + f(\dot{z}) + kz = \frac{Q\rho}{g}\omega^2 \sin \omega t.$$

Let us denote: $\frac{Q\rho}{Q} = A_\infty; \quad \lambda^2 = \frac{k}{m}; \quad \varphi(\dot{z}) = \frac{f(\dot{z})}{m}$

and rewrite equation (1):

$$\ddot{z} + \varphi(\dot{z}) + \lambda^2 z = A_\infty \omega^2 \sin \omega t. \qquad (19)$$

There is no precise solution of this equation. For an approximate solution use is made of the equivalent equation with linear attenuation[3]:

$$\ddot{z} + 2n\dot{z} + \lambda^2 z = A_\infty \omega^2 \sin \omega t. \qquad (20)$$

where

$$2n = \frac{a}{m}; \quad \lambda^2 = \frac{k}{m}.$$

The solution of this equation is as follows:

$$\left.\begin{aligned} z = A\sin(\omega t - \psi); \quad A = \frac{A_\infty \omega^2}{\sqrt{(\lambda^2 - \omega^2)^2 + 4n^2\omega^2}}; \\ \operatorname{tg}\psi = \frac{2n\omega}{\lambda^2 - \omega^2}. \end{aligned}\right\} \qquad (21)$$

We also assume that the work of damping forces within a vibration cycle in equations (19) and (20) is the same, i.e.,

$$W=\int_0^T f(\dot z)\,dz=\int_0^T f(\dot z)\,\dot z\,dt=\int_0^T \alpha\,\dot z^2\,dt.$$

On calculating the integrals for a vibration cycle, we evidently have

$$\int_0^{2\pi}=4\int_0^{\frac{\pi}{2}}$$

and

$$4\int_0^{\frac{\pi}{2}}\cos^{p+1}\omega t\,d(\omega t)=2\sqrt{\pi}\,\frac{\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+3}{2}\right)}.$$

If we take $f(\dot z) = \beta\dot z^p$ (the sign of β must be such that the friction force would counteract the movement), then on substituting into

$$\int_0^T \alpha\,\dot z^2\,dt=\int_0^T \beta\,\dot z^{p+1}\,dt,$$

we obtain

$$\alpha A^2\omega^2\pi=\beta\omega^{p+1}A^{p+1}\,2\sqrt{\pi}\,\frac{\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+3}{2}\right)}.$$

or

$$\frac{\alpha}{m}=(A\omega)^{p-1}\gamma\frac{\beta}{m}, \qquad (22)$$

where

$$\gamma=\frac{2}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+3}{2}\right)}. \qquad (23)$$

The values of coefficient γ are given below.

n	0	1/2	1	2	3	4	5	6
γ	$\frac{4}{\pi}$	1,11284	1	$\frac{8}{3\pi}$	$\frac{3}{4}$	$\frac{32}{15\pi}$	$\frac{5}{8}$	$\frac{64}{35\pi}$

On substituting the values of $\frac{\alpha}{m}$ from (22) into (21), we obtain:

$$A = \frac{A_{\infty}\omega^2}{\sqrt{(\lambda^2-\omega^2)^2 + 4l^2\gamma^2(A\omega)^{2p-2}\omega^2}}, \tag{24}$$

where

$$2l = \frac{\beta}{m}; \quad 2n = \frac{\alpha}{m}. \tag{25}$$

After simple algebraic transformations we obtain from equation (10):

$$A^2(\lambda^2-\omega^2)^2 + 4l^2\gamma^2 A^{2p}\omega^{2p} = A_{\infty}^2\omega^4. \tag{26}$$

At $p = \frac{1}{2}$, on denoting

$$\zeta = \frac{2l^2\gamma^2}{\lambda^3 A_{\infty}}, \quad \bar{\omega} = \frac{\omega}{\lambda},$$

we have

$$\frac{A}{A_{\infty}} = \frac{-\zeta\bar{\omega}}{(1-\bar{\omega}^2)^2} \pm \sqrt{\frac{\zeta^2\bar{\omega}^2}{(1-\bar{\omega}^2)^4} + \frac{\bar{\omega}^4}{(1-\bar{\omega}^2)^2}}.$$

From equation (26) at $\omega = \lambda$, we obtain

$$\frac{A}{A_{\infty}} = \frac{A_{\infty}\omega^3}{4l^2\gamma^2} = \frac{1}{2\zeta}.$$

At $p = 2$, on denoting $\zeta = 4l^2\gamma^2 A^2_{\infty}$, we have

$$\left(\frac{A}{A_\infty}\right)^2 = \frac{-(1-\bar{\omega}^2)^2}{2\zeta\bar{\omega}^4} \pm \sqrt{\frac{(1-\bar{\omega}^2)^4}{4\zeta^2\bar{\omega}^8} + \frac{1}{\zeta}}.$$

Figure 3 shows the resonance curves for two values of p:

$$p = \frac{1}{2} \text{ and } p = 2$$

Let us examine a combination of dry and viscous friction.

A precise solution was given by Den Hartog[(4)]. However, we cannot use his resonance curves because in our case the perturbation force is proportional to ω^2. Since the differential equation is nonlinear, there will be no simple proportionality between the perturbation force and the amplitude. Therefore, all curves have been recalculated. Den Hartog's conclusions are briefly as follows. In the presence of dry friction, we should distinguish between motion without stops (Figure 4a) and motion with one or several stops (Figure 4b). The stops result from the fact that the perturbation force cannot overcome the friction force and the bearing capacity of soil.

In the case of simple motion without stops, the differential vibration equation may be expressed in conventional terms throughout the entire period of time:

$$m\ddot{z} + kz \pm F + \alpha\dot{z} = \frac{Q_{0\varepsilon}}{g}\,\omega^2\cos(\omega t + \varphi). \qquad (27)$$

In the half-period $0<t<\frac{\pi}{\omega}$ the velocity is negative, and therefore for this interval F must have a minus sign.

The phase shift between the perturbation force and the friction force is defined by angle ϕ calculated from boundary conditions which will be discussed later. Let us divide the equation by $m = \frac{Q}{g}$ and introduce the following terms:

$$\frac{k}{m} = \lambda^2; \quad \frac{F}{m} = \frac{F}{k}\lambda^2 = f\lambda^2; \quad A_\infty = \frac{Q_{0\varepsilon}}{Q}; \quad \frac{\alpha}{m} = 2n.$$

Then within the interval $0<t<\frac{\pi}{\omega}$ equation (27) will have the following form:

$$\ddot{z}+\lambda^2(z-f)+2n\dot{z}=A_\infty\omega^2\cos(\omega t+\varphi). \tag{29}$$

Let

$$p=\sqrt{\lambda^2-n^2};\quad q=\frac{1}{\lambda^2}\sqrt{(\lambda^2-\omega^2)^2+4n^2\omega^2}; \tag{30}$$

$$q\lambda^2\cos\varepsilon=(\lambda^2-\omega^2)\cos\varphi+2n\omega\sin\varphi;$$

$$q\lambda^2\sin\varepsilon=(\lambda^2-\omega^2)\sin\varphi-2n\omega\cos\varphi.$$

The general solution of this differential equation is as follows:

$$z=e^{-nt}(C_1\cos pt+C_2\sin pt)+\frac{A_\infty\omega^2}{q\lambda^2}\cos(\omega t+\varepsilon)+f. \tag{31}$$

The three arbitrary constants C_1, C_2 and ϕ are found from four boundary conditions including one other unknown A which, consequently, is also found from these equations at $t = 0$, $z = A$, $z = 0$ and

$$t=\frac{\pi}{\omega},\quad z=-A,\quad \dot{z}=0.$$

Without going into detail, let us represent the solution of this equation in a form which differs somewhat from that used by Den Hartog:

$$\frac{A}{A_\infty}=\sqrt{\frac{1}{q^2}\left(\frac{\omega}{\lambda}\right)^4-\left(\frac{f}{A_\infty}\right)^2H^2}-\frac{f}{A_\infty}G, \tag{32}$$

where

$$H=\frac{\lambda^2}{p\omega}\cdot\frac{\sin\frac{p\pi}{\omega}}{\operatorname{ch}\frac{n\pi}{\omega}+\cos\frac{p\pi}{\omega}};$$

$$G=\frac{\operatorname{sh}\frac{n\pi}{\omega}-\frac{n}{p}\sin\frac{p\pi}{\omega}}{\operatorname{ch}\frac{n\pi}{\omega}+\cos\frac{p\pi}{\omega}};$$

$$\sin\varepsilon=\frac{-q\lambda^2}{A_\infty\omega^2}fH;\quad \cos\varepsilon=\frac{q\lambda^2}{A_\infty\omega^2}(A+Gf). \tag{33}$$

$\sin\phi$ and $\cos\phi$ are found from (30).

Let us now examine the case of motion with one stop. We shall assume that the stop starts at $t = t_0$.

The differential equation for the section from 0 to t_0 has the same form (29) as for the case of continuous motion. Therefore, its general solution is also given by equation (31).

The boundary conditions are similar:

$$\text{at } t = 0,\ z = A,\ \mathbf{z} = 0,$$

$$\text{at } t = t_0,\ \dot{z} = -A,\ \dot{z} = 0.$$

In contrast to the preceding case, the equation will contain the unknown factor t_0. The additional ratio for its determination may be obtained if we observe what happens at the beginning of motion $t = 0$.

Since at this moment the mass is at rest, velocity and acceleration are equal to zero, i.e., $\dot{z} = 0$ and $\ddot{z} = 0$.

In this case the differential equation (27) has the following form:

$$kA - F - \frac{Q_0 e}{g}\,\omega^2 \cos\varphi = 0$$

or

$$A - f = \frac{A_\infty \omega^2}{\lambda^2} \cos\varphi. \qquad (35)$$

All unknowns are found from equations (34) and (35):

$$C_1;\quad C_2;\quad \Phi;\quad A \text{ and } t_0.$$

We shall omit the intermediate calculations and proceed directly to the solution of the differential equations for the section $0 \leqslant t \leqslant t_0$:

$$\frac{A}{A_\infty} = \frac{f}{A_\infty} + \frac{\omega^2}{\lambda^2}\cos\varphi;$$

$$\sin\varphi = \frac{Aq^2}{\sqrt{(Aq^2)^2 + (Bq^2)^2}}; \quad \cos\varphi = \frac{Bq^2}{\sqrt{(Aq^2)^2 + (Bq^2)^2}};$$

$$Aq^2 = \left(1 - \frac{\omega^2}{\lambda^2}\right)e^{nt_0}\sin\omega t_0 - \frac{2n\omega}{\lambda^2}e^{nt_0}\cos\omega t_0 + \frac{2n\omega}{\lambda^2}\cos pt_0 + \tag{36}$$

$$+ \frac{\omega}{p}\left(\frac{2n^2}{\lambda^2} + \frac{\omega^2}{\lambda^2} - 1\right)\sin pt_0;$$

$$Bq^2 = \frac{-2n\omega}{\lambda^2}e^{nt_0}\sin\omega t_0 - \left(1 - \frac{\omega^2}{\lambda^2}\right)e^{nt_0}\cos\omega t_0 +$$

$$+ \left(1 - \frac{\omega^2}{\lambda^2}\right)\cos pt_0 + \frac{n}{p}\left(1 + \frac{\omega^2}{\lambda^2}\right)\sin pt_0.$$

$\cos\varepsilon$ and $\sin\varepsilon$ are found from (30) and t_0 from the transcendental equation:

$$\frac{2f\lambda^2}{A_\infty\omega^2} = -\cos\varphi + e^{-nt_0}\cos pt_0\left(\frac{\cos\varepsilon}{q} - \cos\varphi\right) +$$

$$+ e^{-nt_0}\sin pt_0\left(\frac{n}{pq}\cos\varepsilon - \frac{n}{p}\cos\varphi - \frac{\omega}{pq}\sin\varepsilon\right) - \frac{1}{q}\cos(\omega t_0 + \varepsilon). \tag{37}$$

This equation cannot be solved directly with respect to t_0 at given f and ω. It can be solved by the graphoanalytical method. If t_0 and ω are given, we can find A and f analytically from (35) and (37). Then for a given ω we can construct two sets of curves (A, t_0) and (f, t_0) from which we can find t_0 for the given A and f. At $t_0 = \frac{\pi}{\omega}$ we obtain the boundary between the two solutions.

Figure 5 shows the resonance curves for one case of viscous friction $\frac{n}{\lambda} = 0.5$ and different values of dry friction f calculated from precise equations. The dotted line in Figure 13* represents the boundary separating the region of motion with one step from the region of continuous motion.

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* Sic. Probably Figure 5 (Transl.).

The curves in Figure 6 illustrate the case where viscous friction is absent, i.e., $\frac{n}{\lambda} = 0$.

The continuous resonance curves have been constructed from approximate equations. The dotted lines and individual points indicate the points of resonance curves calculated from precise equations.

The form of the approximate solution is simple, and closely resembles that of the precise equation:

$$\frac{A}{A_\infty} = \frac{\sqrt{\left(\frac{\omega}{\lambda}\right)^4 - \left(\frac{4}{\pi} \cdot \frac{f}{A_\infty}\right)^2}}{1 - \left(\frac{\omega}{\lambda}\right)^2}. \tag{38}$$

<u>Third case. Forced vibrations of a pile with separation from the soil</u>.

We shall examine the simplest case where the action of soil on the pile is replaced by the reaction of a spring (inertialess) with rigidity c, while the mass is assumed to be concentrated at a point.

Let x_1 denote the position of this point during its motion on separation from the soil (spring), and x_2 denote the position when the point is on the soil (spring). Let the boundary between these positions ($x_1 = x_2$) be the start of the coordinates.

Let us assume that steady-state motion with a period which is a multiple of the period of induced force ($nT = \frac{2\pi n}{\omega}$, where ω is the frequency of the induced force) is possible.

As is usually done in the case of vibrators, let us represent the induced force as follows:

$$P = \frac{Q_0 \varepsilon}{g} \omega^2 \sin(\omega t + \varphi),$$

where Q_0 is the driving moment of the vibrator and ϕ is the as yet unknown phase shift between the induced force and the separation moment.

Let us assume that in the time interval $0 \leqslant t \leqslant t_1$ the pile moves while separated from the soil ($t = 0$ is the separation moment, $t = t_1$ is the moment of impact with the spring). If Q is the weight of pile, then the differential equation of its motion in this time integral will be as follows:

$$\frac{Q}{g}\ddot{x}_1 = \frac{Q_0\varepsilon}{g}\omega^2 \sin(\omega t + \varphi) + Q; \quad x_1 \leqslant 0. \tag{39}$$

Let us assume further that at $t_1 \leqslant t \leqslant \frac{2\pi n}{\omega}$ the pile vibrates together with the spring*. The differential equation of its motion in this time interval will have the following form:

$$\frac{Q}{g}\ddot{x}_2 + cx_2 = \frac{Q_0\varepsilon}{g}\omega^2 \sin(\omega t + \varphi) + Q; \quad x_2 > 0. \tag{40}$$

The differential equations (39) and (40) must be integrated at the following boundary conditions:

$$\left.\begin{array}{c} \text{at} \quad t = 0, \quad x_1 = 0, \\ \text{at } t = t_1, \quad x_1 = x_2 = 0, \quad x_1'(t_1) = x_2'(t_1), \\ \text{at} \quad t = \frac{2\pi n}{\omega}, \quad x_2 = 0 \\ \text{and at} \quad x_1'(0) = x_2'\left(2\frac{\pi n}{\omega}\right). \end{array}\right\} \tag{41}$$

It is assumed under such conditions that the fall of the pile to the ground occurs without sudden changes in the rate of fall.

Hence, altogether there are six boundary conditions.

Let

$$A_\infty = \frac{Q_0\varepsilon}{Q}, \quad k = \frac{\lambda^2 A_\infty}{g}, \quad \zeta = \frac{\lambda}{\omega}. \tag{42}$$

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* We could have assumed in a more general case that the pile would separate from the soil more than once within this period. However, it was found experimentally that the separation occurs within the first, second and third periods of the induced force.

Then a general solution of equations (39) and (40) will have the following form:

$$\frac{x_1}{A_\infty} = -\sin(\omega t + \varphi) + \frac{gt^2}{2A_\infty} + \frac{at}{A_\infty} + \frac{b}{A_\infty}, \quad 0 \leqslant t \leqslant t_1$$

$$\frac{x_2}{A_\infty} = -\frac{1}{1-\zeta^2}\sin(\omega t + \varphi) + \frac{1}{k} + \frac{c}{A_\infty}\sin\lambda t + \frac{d}{A_\infty}\cos\lambda t, \qquad (43)$$

$$t_1 \leqslant t \leqslant \frac{2\pi n}{\omega},$$

where

$$x_1 \leqslant 0, \quad x_2 > 0.$$

The six unknowns (a, b, c, d, ϕ, and t_1) are found from the boundary conditions given above.

The unknown time t_1 is determined from the following transcendental equation:

$$k = \frac{\left(\frac{\eta\tau}{2} + \operatorname{tg}\frac{\alpha}{2}\right)(1-\zeta^2)}{\zeta\cos\varphi + \sin\varphi \operatorname{tg}\frac{\alpha}{2}}. \qquad (44)$$

where

$$t_1 = \frac{2\pi n\tau}{\omega}; \quad \eta = 2\pi n\zeta; \quad \alpha = \eta(1-\tau);$$

$$\lambda t_1 = \eta\tau = \zeta\omega t_1; \quad \omega t_1 = 2\pi n\tau.$$

For ϕ we have the following expression:

$$\varphi = \frac{\pi}{2} - \frac{\omega t_1}{2} \text{ or } \varphi = \frac{3\pi}{2} - \frac{\omega t_1}{2},$$

Here the value of ϕ is such that k is positive. Finally:

$$\frac{a}{A_\infty \omega} = \frac{-\eta\tau\zeta}{2k}; \quad \frac{b}{A_\infty} = \sin\varphi; \quad \frac{c}{A_\infty} = M\left(\sin\eta - \operatorname{tg}\frac{\alpha}{2}\cos\eta\right);$$

$$\frac{d}{A_\infty} = M\left(\cos\eta + \operatorname{tg}\frac{\alpha}{2}\sin\eta\right); \qquad M = \frac{\frac{\eta\tau}{2}\operatorname{tg}\varphi - \zeta}{k\left(\operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\varphi + \zeta\right)}.$$

The pile velocity at the moment of separation from the soil ($t = 0$ or $t = \frac{2\pi n}{\omega}$) is given by:

$$\frac{v}{A_\infty \omega} = -\cos\varphi - \frac{\tau_1 \alpha}{2k}.$$

The pile velocity at the moment of impact with the soil ($t = t_1$) is equal to but has the opposite sign than the velocity at the moment of separation.

The physical meaning of the problem evidently demands that

$$\left(-\frac{v}{A_\infty \omega}\right)_{t=t_1} > 0.$$

If this condition is not satisfied, then the solutions are nonsensical, although they satisfy all boundary conditions and differential equations.

It is readily seen that x_1 and x_2 in the solution obtained will be symmetrical with respect to the middle of their sections.

The investigation of the solution obtained is made difficult by the transcendence of the equation.

The resonance curves should be constructed by taking k for a number of values of ζ to find τ, which does not have to be single-valued, and then by using equation (5) construct the curve of corresponding motion, from which the vibration amplitude could be determined.

If k is given, τ may be found from the k curve.

Figure 7 shows the $k(\zeta)$ curves for various values of τ and two values of n. The dotted sections correspond to $\left(\frac{v}{A_\infty \omega}\right)$ $t = t_1 < 0$. Consequently, the solutions for these sections have no physical meaning.

As may be seen from the curves, k for certain n and ζ is a very complex function of τ.

Figure 8 shows the curves of pile motion under different conditions. It may be seen that there are large variations in the form of pile motion.

3. Comparison of Experimental and Theoretical Resonance Curves

The circles on curves in Figure 9 indicate the experimental data. The frequencies $\omega = 0.014\ N$ (N - rpm of the vibrator) are shown along the abscissa, while the relative amplitudes $\frac{A}{A_\infty}$ are shown along the ordinate.

On comparing experimental and theoretical data we may conclude that the experimental data are best approximated by theoretical curves with allowances for the highest terms of expansion in the bearing capacity of soil and introducing one more coefficient a_∞.

The experimental values of amplitudes for high driving frequencies of the vibrator tend towards values less than the theoretical $\left(\frac{A}{A_\infty} = 1\right)$. Therefore, to express the experimental data by way of theoretical resonance curves, it is essential to introduce one more coefficient (a_∞), which represents the actual ultimate value of relative amplitudes on the increase in rpm. The decrease in the ultimate value of A_∞ is probably due to the fact that some soil in contact with the pile is subjected to vibration; consequently, the actual weight of the pile must be increased.

Figures 9a, b, c, d and e show two theoretical curves for $u = 0$, (dotted lines) and $u \neq 0$ (continuous lines).

For the driving moment of 660 kgcm, we take $u = 0.5$ as the closest to experimental data. Since $u = \frac{3\gamma}{4} \cdot \frac{A^2_\infty}{\lambda^2}$, while γ and λ for the given soil conditions are approximately constant, then for the driving moments 495, 330 and 165 kgcm we should take $u = 0.30$, 0.10 and 0 respectively.

The curves show that λ is greater at $u \neq 0$ than at $u = 0$. This is due to the fact that in the linear theory the bearing capacity of soil is given in the form of one term kz, while in the nonlinear theory it is represented by three terms:

$$kz - dz^3 + ez^5 = \lambda^2 m A_\infty \left[\frac{z}{A_\infty} - \frac{4u}{3}\left(\frac{z}{A_\infty}\right)^3 + \frac{8v}{5}\left(\frac{z}{A_\infty}\right)^5\right].$$

Therefore, the linear theory gives an averaged value of k (and consequently λ^2), which is less than the calculated first term in the nonlinear theory.

The deviation of theoretical resonance curves from experimental values is often greatest at low frequencies, especially at u = 0. This is evidently due to the fact that in the case of theoretical curves the friction forces are taken as proportional to the vibration rate. This law probably holds better for large frequencies than for small frequencies.

At small driving moments (Figure 9), the amplitudes on the resonance curve in the resonance region differ greatly from the ultimate values. At large eccentric moments the resonance is not evident.

Tables I-VII contain the parameters of theoretical resonance curves for testing piles and sheet piles.

The tables contain two values of u, λ, $\frac{n}{\lambda}$ and a_∞ for a number of piles and four eccentric moments.

The data in the tables show that at driving moments 660, 495 and 330 kgcm and u = 0, $\frac{n}{\lambda}$ (the damping coefficient), λ sec $^{-1}$ (natural frequency) and a_∞ (the coefficient of lowering the ultimate amplitude) are in some cases approximately the same. This indicates that for the given range of the eccentric moment, A is linearly dependent on $A_\infty = \frac{Q_0\varepsilon}{Q}$, and consequently on the eccentric moment $Q_0\varepsilon$, mainly at $\omega > \lambda$.

The power used by the vibrator motor was recorded simultaneously with the recording of the resonance curves.

Figure 10 shows the power $W-W_0$ vs. the square of vibration rate $(A\omega)^2$. W_0 is the power lost in the vibrator itself, which was measured in the vibrator freely suspended on a shock absorber. The results of measurements are given in Figure 11.

It follows from (18) that

$$\frac{W-W_0}{(A_\infty)^2} = \frac{Qn10^{-4}}{g} \approx Qn10^{-7} \text{ kg sec/cm} \qquad (45)$$

The friction coefficient n calculated from the resonance curves does not quite coincide with that calculated from (39), as may be seen from Figure 9 which shows straight lines* at coefficients n obtained from the resonance curves (at eccentric moments 660, 495 and 330 kg cm).

For wooden piles, n determined from the power measurements will be somewhat smaller, while for a sheet pile, somewhat larger than that found from the resonance curves.

This is probably due to the fact that n is not a constant, but depends on the number of revolutions N. Moreover, it would be more accurate to determine W_0 while the vibrator is firmly fixed.

Conclusions

1. At large values of eccentric moments of the vibrator, the amplitude of forced vertical vibrations of piles and sheet piles may be calculated mainly from the nonlinear vibration theory.

At small eccentric moments, the amplitudes are well approximated by the linear theory ($u = 0$, $v = 0$).

In the case of vibration frequencies higher than the natural frequencies, the amplitudes may be calculated from the linear theory irrespective of the magnitude of the driving moment.

2. The ultimate values of amplitudes A_∞ differ from the theoretical values. This may be corrected by using coefficient a_∞, which for the given soil conditions varies between 0.60 and 0.80.

3. The natural frequencies of wooden piles and metallic sheet pile are somewhat dependent on their cross-sections and the depth of installation. For the given soil conditions, the natural vibration frequencies vary between 60 and 85 sec^{-1} according to the non-linear theory, and from 55 to 70 sec^{-1} if calculated from the linear theory.

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* Sic. "Continuous lines"? (Transl.).

4. The power required to maintain forced vibrations of a pile is very approximately given by equation (18). For practical purposes, the coefficient of friction n may be taken as approximately equal to 20-30 sec^{-1}.

5. The total power required by the vibrator motor is greatly affected by the power required to overcome the resistance in the vibrator itself.

6. It is essential to carry out tests in various types of soil to determine the effect of soil properties on the forced vibrations of piles.

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2. S. P. Timoshenko Teoriya kolebaniya v inzhenernom dele (Vibration theory in engineering), GNTI, 1932.
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Table I

Wooden pile, d = 12 cm, weight* 735 kg

Driving moment in kgcm / Parameters	660		495		330		165
u	0,50	0	0,30	0	0,1	0	0
n/λ	0,30	0,40	0,35	0,40	0,35	0,40	0,20
λ sec^{-1}	80	65	72	62	73	65	65
a_{∞}	0,74	0,74	0,71	0,66	0,72	0,72	0,54

Table II

Wooden pile, d = 15 cm, weight 760 kg

Driving moment in kgcm / Parameters	660		495		330		165
u	0,50	0	0,30	0	0,1	0	0
n/λ	0,30	0,40	0,35	0,40	0,35	0,35	0,20
λ sec.^{1}	75	58	85	65	65	65	65
a_{∞}	0,67	0,67	0,76	0,72	0,66	0,61	0,49

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* Including vibrator

Table III

Wooden pile, d = 20 cm, weight 800 kg

Driving moment in kgcm / Parameters	660		495		330		165
u	0,50	0	0,30	0	0,10	0	—
n/λ	0,30	0,40	0,35	0,40	0,35	0,40	—
λ sec.$^{-1}$	75	58	80	65	65	58	—
a_∞	0,57	0,64	0,72	0,70	0,64	0,67	—

Table IV

Wooden pile, d = 25 cm, weight 860 kg

Driving moment in kgcm / Parameters	660		495	330	165
u	0,50	0	0	0	0
n/λ	0,30	0,40	0,30	0,35	0,20
λ sec.$^{-1}$	60	50	50	50	65
a_∞	0,78	0,78	0,68	0,84	0,57

Table V

Wooden pile, d = 31 cm, weight 890 kg

Driving moment in kgcm / Parameters	660	495	830	165
u	0,50	0	0	—
n/λ	0,30	0,30	0,35	—
λ sec.$^{-1}$	65	57	5[illegible]	—
a_∞	0,84	0,78	0,84	—

Table VI

Flat ShP-type sheet pile, 890 kg

Driving moment in kgcm / Parameters	660		495		330		166
u	0,5	0	0,30	0	0,1	0	0
n/λ	0,40	0,50	0,40	0,50	0,40	0,50	0,30
λ sec.$^{-1}$	60	55	58	55	65	65	77
a_∞	0,66	0,66	0,70	0,74	0,72	0,78	0,48

Table VII

Trough-like ShK-type sheet pile, 905 kg

Parameters \ Driving moment in kgcm	660		495		330		165
a	0,50	0	0,30	0	0,10	0	0
n/λ	0,30	0,40	0,35	0,40	0,35	0,40	0,30
λ sec.$^{-1}$	68	55	60	54	66	65	77
a_{∞}	0,65	0,62	0,62	0,62	0,61	0,64	0,72

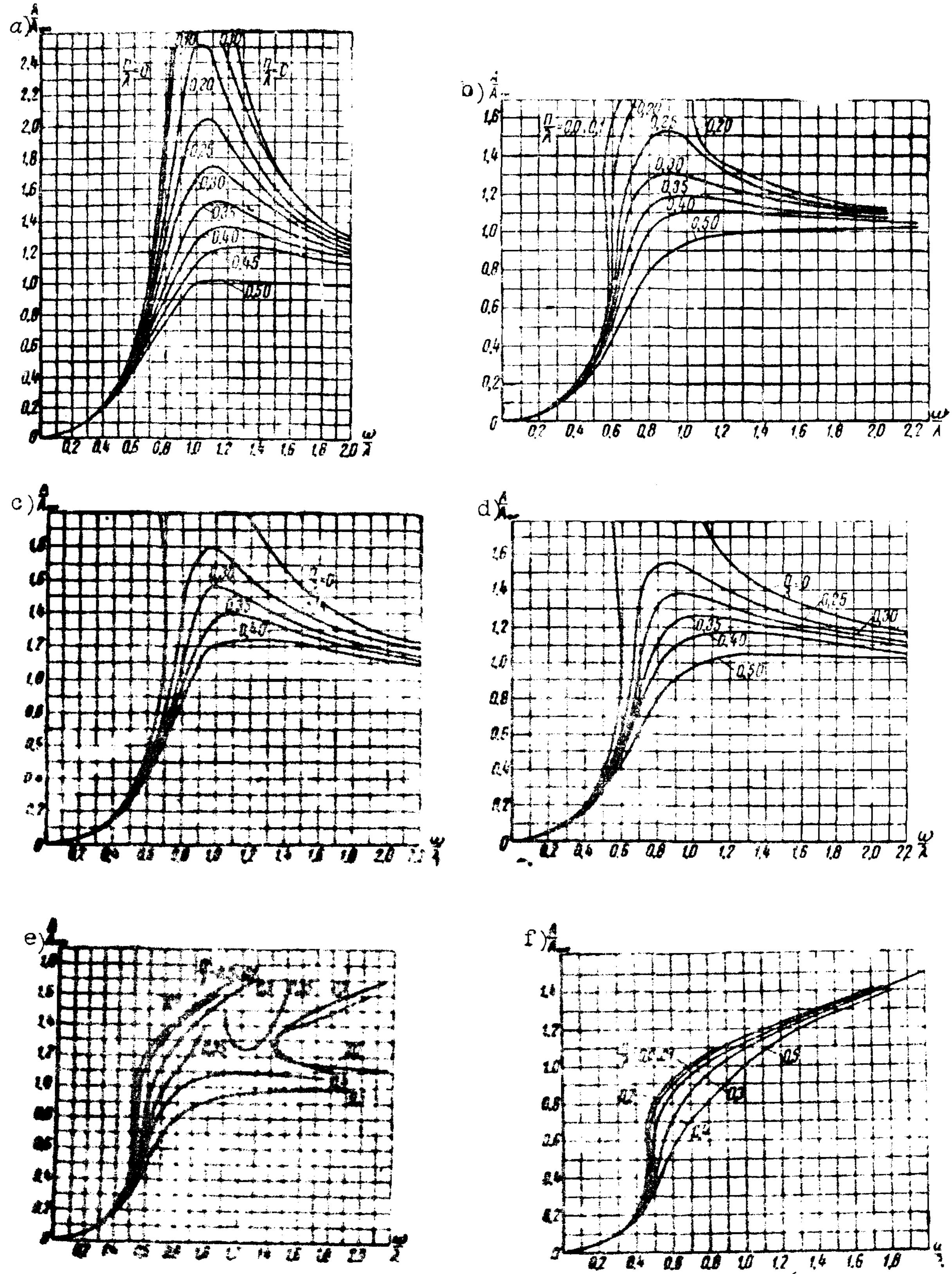

Fig. 1

Resonance curves

a - at u = 0, v = 0; b - at u = 0.5, v = 0.125; c - at u = 0.1, v = 0.005; d - at u = 0.3, v = 0.045; e - at u - 1, v = 0.5; f - at u = 2, v = 2

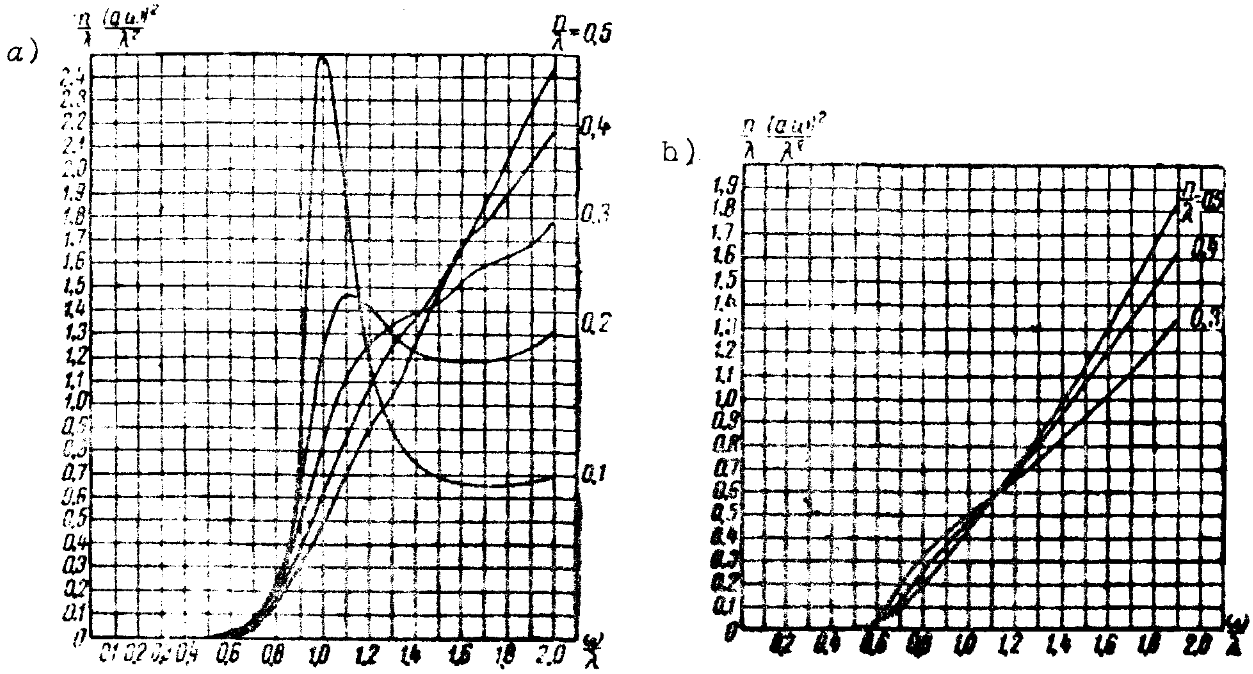

Fig. 2

Power vs. frequency at different damping coefficients

a - at u = 0 (v = 0);
b - at u = 0.5 (v = 0.125)

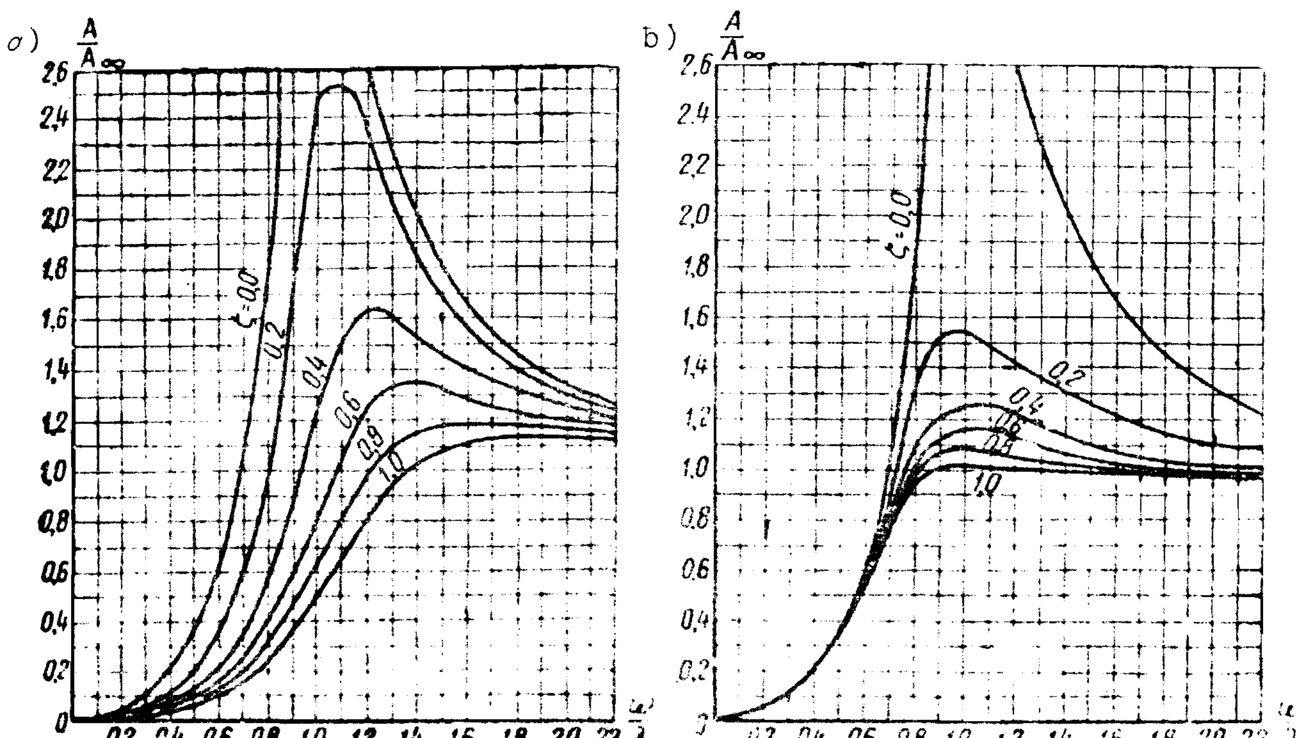

Fig. 3

Resonance curves

a - at $p = \frac{1}{2}$; b - at p = 2

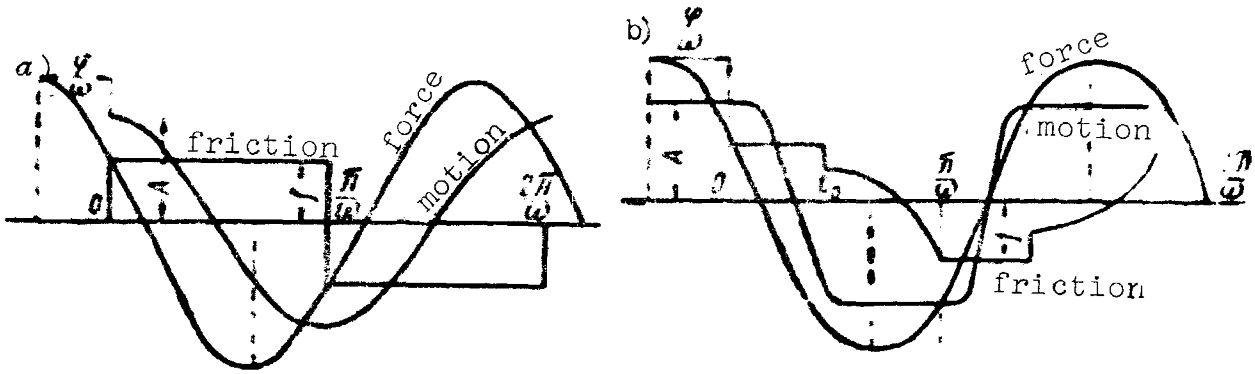

Fig. 4

Motion in the presence of dry friction
(after Den Hartog)

a - continuous motion; b - with one stop

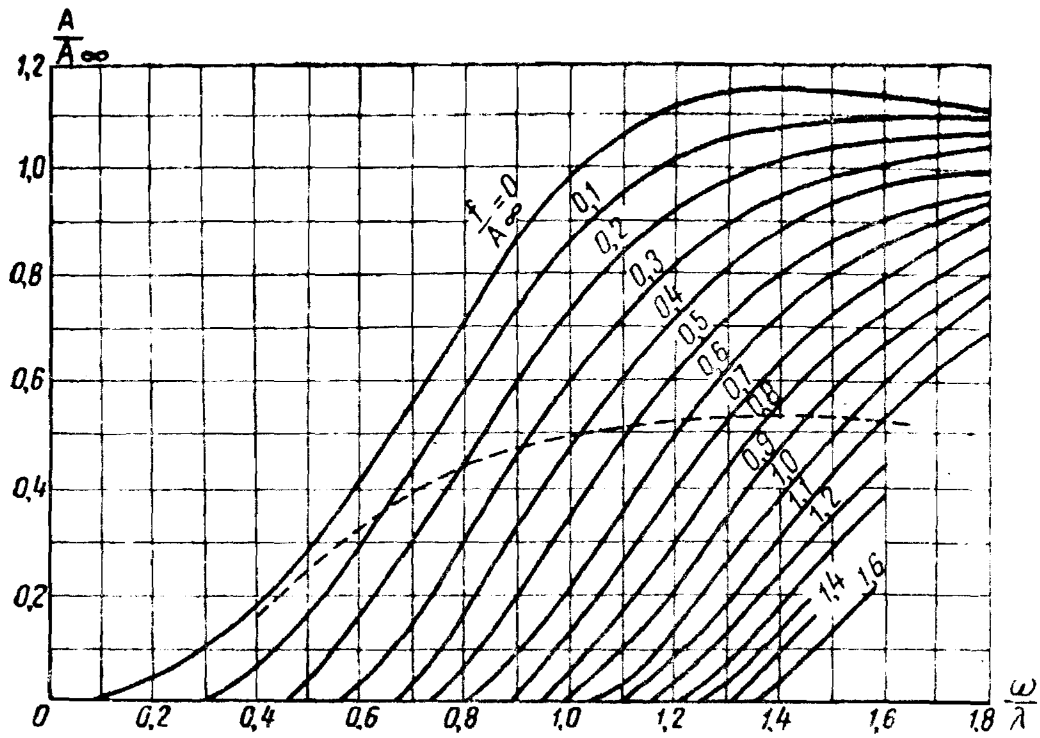

Fig. 5

Resonance curves in the presence of viscous friction

$\frac{n}{\lambda} = 0.5$

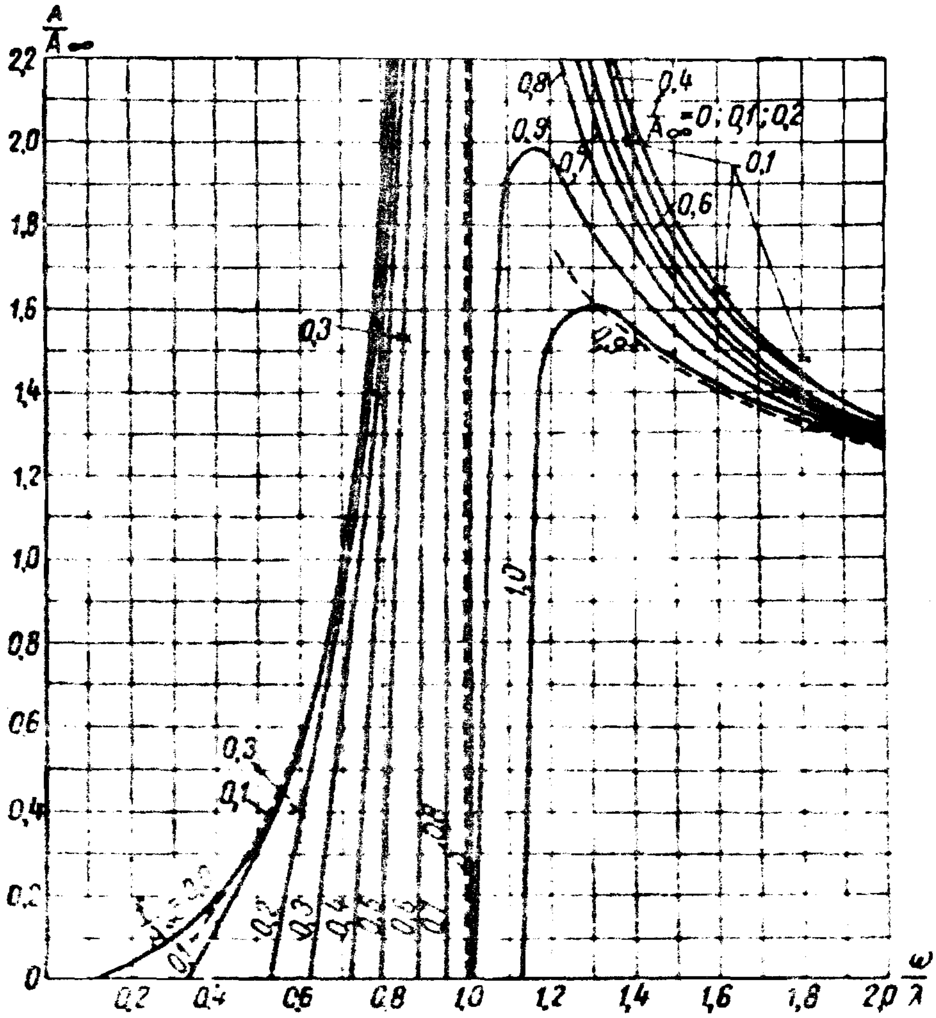

Fig. 6

Resonance curves in the absence of viscous friction

$$\left(\frac{n}{\lambda} = 0\right)$$

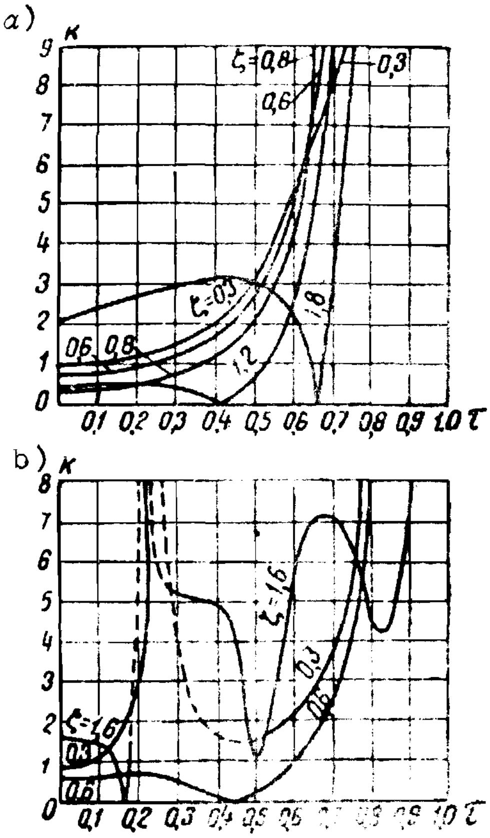

Fig. 7

k = f(τ)

a - at n = 1; b - at n = 2

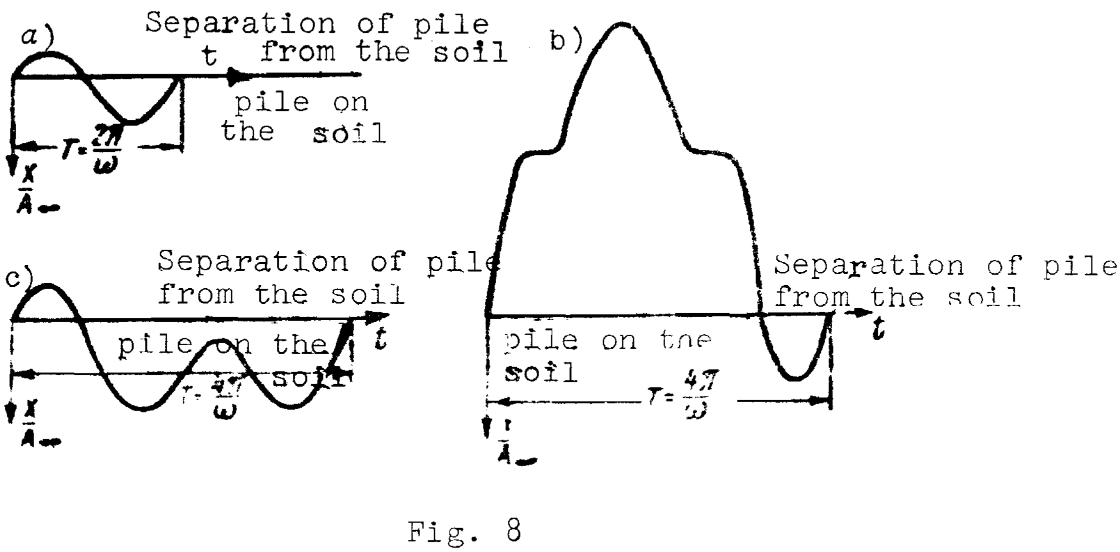

Fig. 8

Pile motion

a - n = 1, ζ = 0.3, τ = 0.4, k = 1.912, φ = 0.314; b - n = 2, ζ = 0.6, τ = 0.2, k = 0.756, φ = 0.314; c - n = 2, ζ = 1.6, τ = 0.8, k = 4.237, φ = -3.456

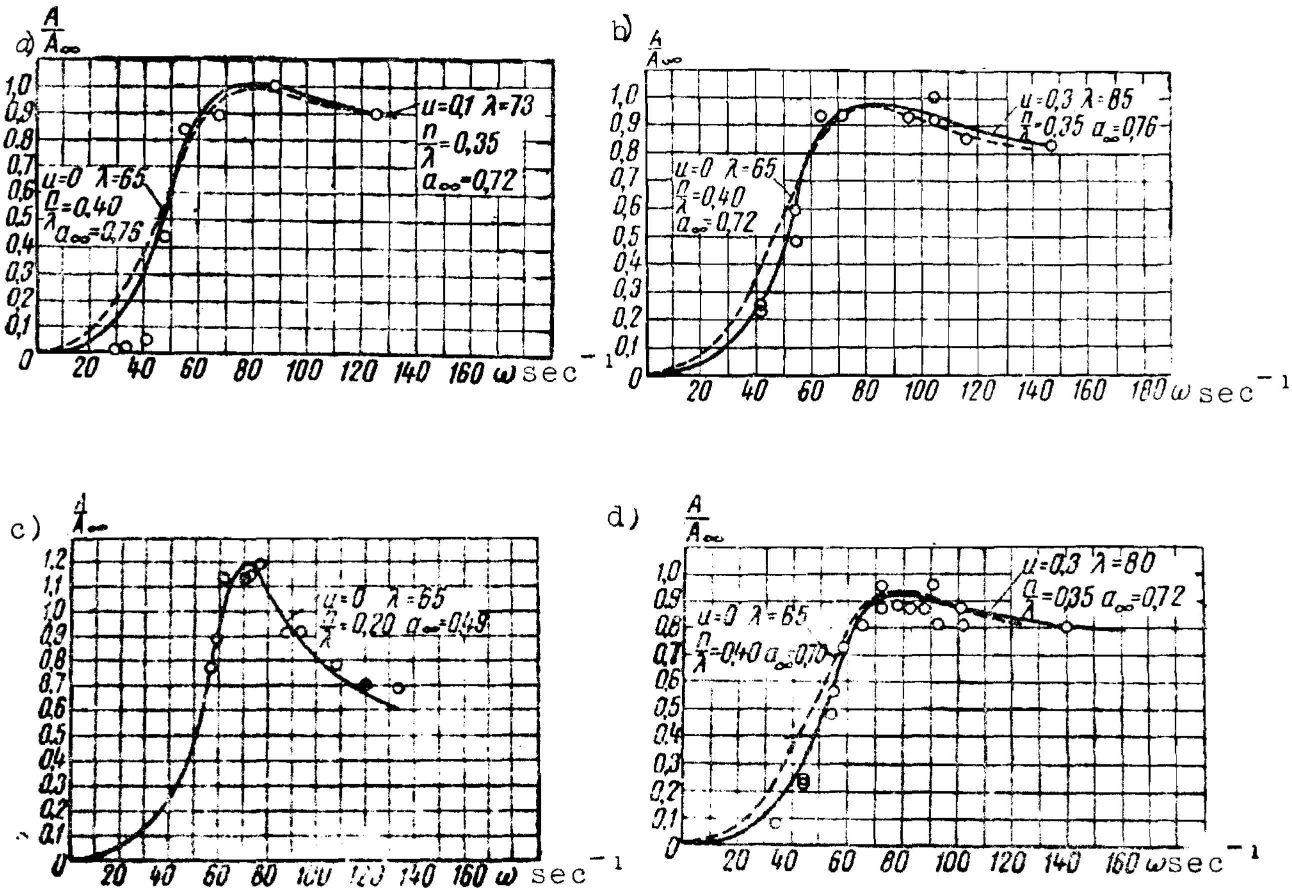

Fig. 9

Resonance curves for:

a - wooden pile d = 12 cm, $A_\infty = \frac{330}{735} = 0.45$ cm;

b - wooden pile d = 15 cm, $A_\infty = \frac{495}{760} = 0.65$ cm;

c - wooden pile d = 15 cm, $A_\infty = \frac{165}{760} = 0.22$ cm;

d - wooden pile d = 20 cm, $A_\infty = \frac{496}{800} = 0.62$ cm;

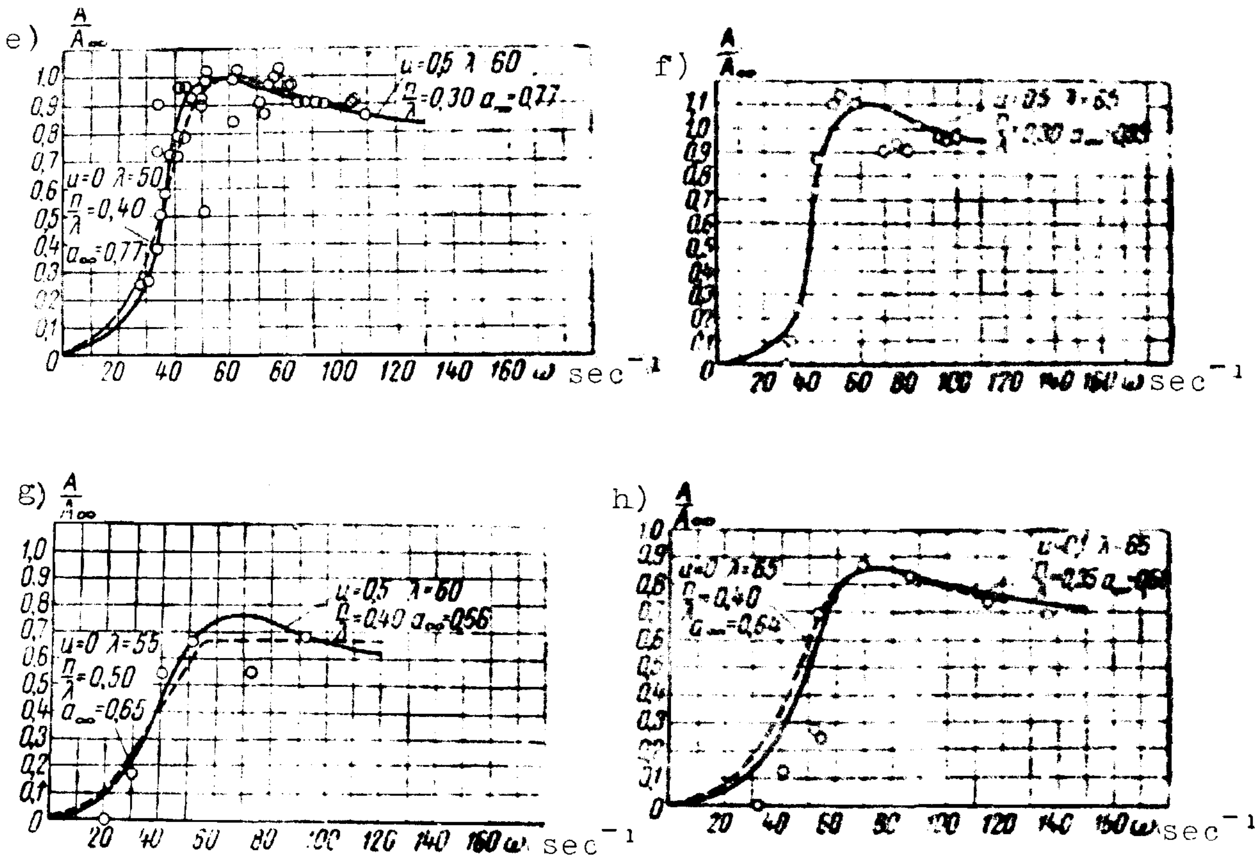

Fig. 9 (Continued)

Resonance curves for:

e - wooden pile d = 25 cm, $A_\infty = \frac{660}{860} = 0.78$ cm;

f - wooden pile d = 31 cm, $A_\infty = \frac{660}{890} = 0.74$ cm;

g - metal ShP sheet pile, $A_\infty = \frac{660}{890} = 0.74$ cm;

h - metal ShK sheet pile, $A_\infty = \frac{330}{905} = 0.36$

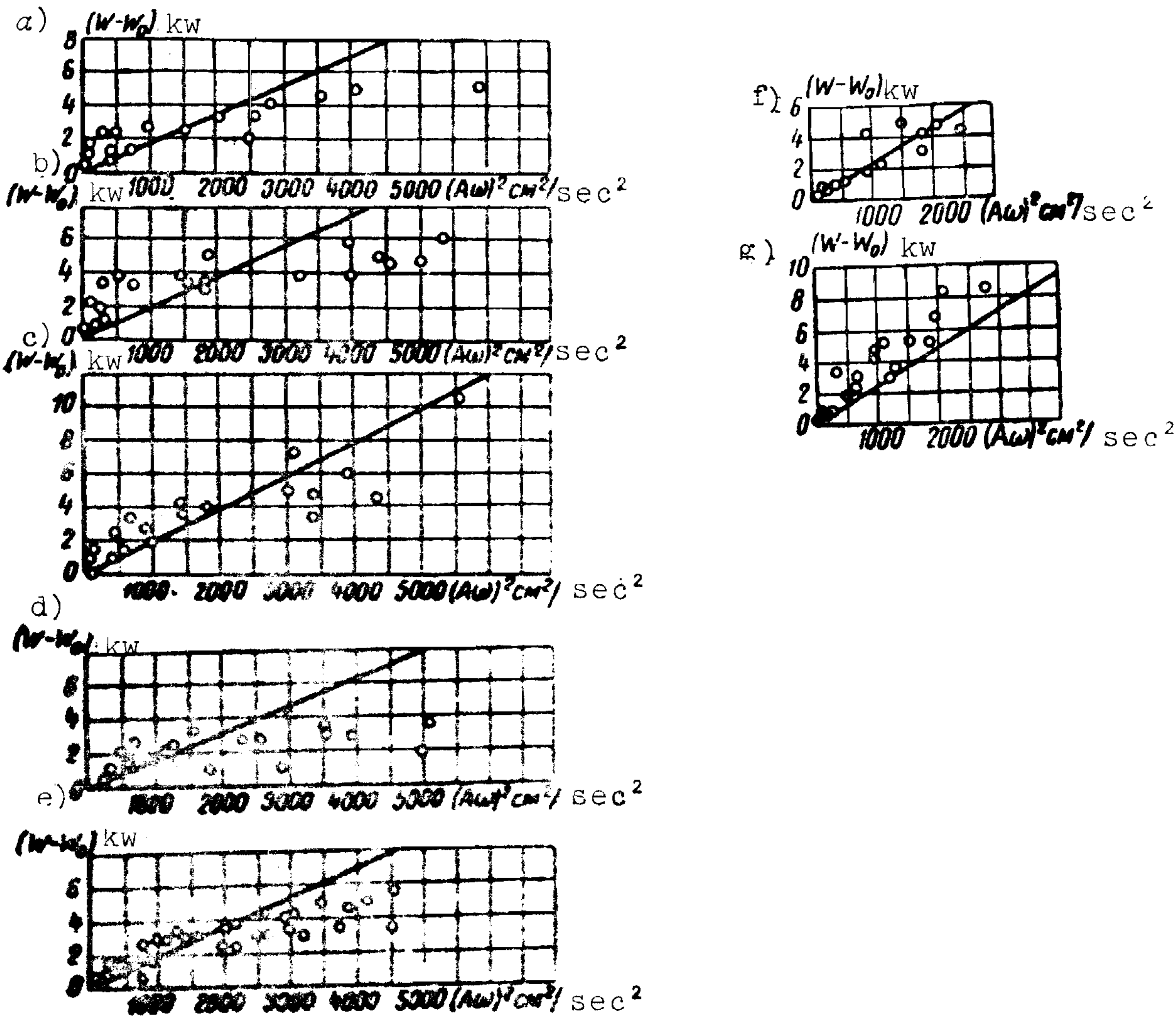

Fig. 10

$$W - W_0 = f\ (A\omega)^2$$

a - wooden pile, d = 12 cm;

b - wooden pile, d = 15 cm;

c - wooden pile, d = 20 cm;

d - wooden pile, d = 25 cm;

e - wooden pile, d = 31 cm;

f - metal ShP sheet pile;

g - metal ShK sheet pile

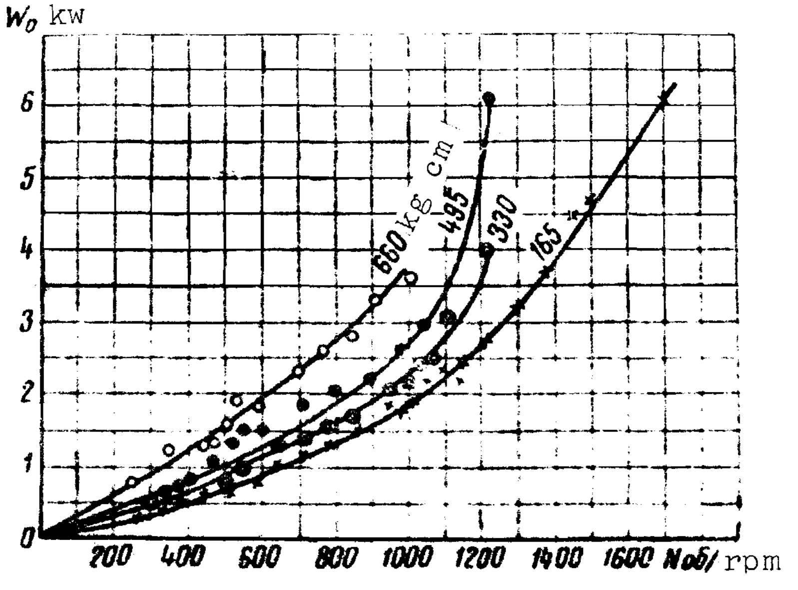

Fig. 11
Power of vibrator freely supported on a shock absorber