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A COHERENT RADAR SYSTEM

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ABSTRACT

The frequency spectrum of the pulse train from a radar transmitter is considered initially, and the spectra of the reflection of this from a fixed and from a moving object are derived.

The received signal is then passed through a coherent detector [i.e., a device that uses a signal "locked" in frequency and phase with the transmitted signal as a means of detection], and the resultant video spectra analysed, again for reflection from a fixed and a moving object. This shows that for the echo from a fixed object, the video signal is a series of constant-amplitude pulses, the amplitude depending upon the cosine of the relative phase angle between the received signal and the detecting signal. For the echo from a moving object, the video signal is a series of pulses varying in amplitude at the "Doppler" frequency for the moving object. Since these pulses vary from positive to negative amplitudes they are termed bipolar pulses.

The video signal is passed through a "3rd detector" which, in effect, integrates it in a series of gates corresponding to range. The behaviour of an ideal integrator is considered and the frequency spectrum of the output derived, again for a "fixed" and a "moving" object. An equivalent of the actual circuit used as the integrator is then considered, and its output compared with that of the ideal integrator. Frequency spectra are again considered, and it is found that the output of this integrating device will drop with a rise in the Doppler frequency by an amount dependent on the time-constant of the integrating circuit. It is also found that the maximum Doppler frequency recognizable is one-half of the pulse repetition frequency.

A device for indicating the direction of motion of an object is also considered.

CONTENTS

A COHERENT RADAR SYSTEM

INTRODUCTION

By a coherent radar system is meant a system whereby the detection of the received signal is accomplished by the use of a signal in coherence with [i.e., locked with] the original transmitted signal. Various methods may be used to accomplish this, such as a master oscillator from which both the detector and transmitter signal are obtained, a "coho" or coherent local oscillator locked by the injection of some of the transmitter signal, and so on.

The main advantages of such a system are twofold. There is an improvement in output signal-to-noise ratio for low-input signal-to-noise ratios,* and it becomes possible to distinguish between the received echoes from a fixed and from a moving target.

Part I of this report is devoted to a consideration of the frequency spectra of the signals involved, from the transmitter down to the video output of the receiver, emphasizing the difference between the echo signals from a fixed and from a moving object. The actual circuitry involved will, however, not be considered.

Part II deals with the results of performing an integration on the video signal, both in an idealized case and as it could be done with a practical circuit.

An integrating circuit, being essentially a very narrow low-pass filter, will enable a further improvement in output signal-to-noise ratio.* However, with a moving object, there are disadvantages which will be considered later.

At this time, it is not proposed to discuss the effect of noise in the system, or the results of using more complicated integrating circuits on the video signal. Experimental evidence confirming the results recorded here exists in a qualitative manner, but it will also be omitted in this report.

PART I: RADIO AND VIDEO FREQUENCY SPECTRA

A. Transmitted Wave Spectrum

A signal, consisting of a series of pulses of amplitude E_o, and width Δt, at a repetition interval of T_r, is used to modulate a carrier

* Smith. J. Inst. Elec. Engrs. (London) IV, vol. 98, p. 43, 1951.

of frequency f_c. The frequency spectrum of this modulating signal is

$$E_m = E_o \left[\frac{\Delta t}{T_r} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \cos 2n\pi \frac{t}{T_r} \right]. \qquad (1.1)$$

This is a series of frequencies spaced by $\frac{1}{T_r}$, with amplitudes as given above [see Fig. 1A].

When this frequency spectrum is used to modulate a carrier of frequency f_c, each modulating frequency n/T_r produces an upper side-band of frequency $f_c + n/T_r$ and a lower sideband of frequency $f_c - n/T_r$ [obtained from the multiplication of the carrier by the modulating signal]. The transmitted wave has, therefore, a frequency spectrum given by

$$E_T = E_o \left[\frac{\Delta t}{T_r} \cos \omega_c t + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos(\omega_c + \frac{2n\pi}{T_r})t \right.\right.$$

$$\left.\left. + \cos(\omega_c - \frac{2n\pi}{T_r})t \right) \right]. \qquad (1.2)$$

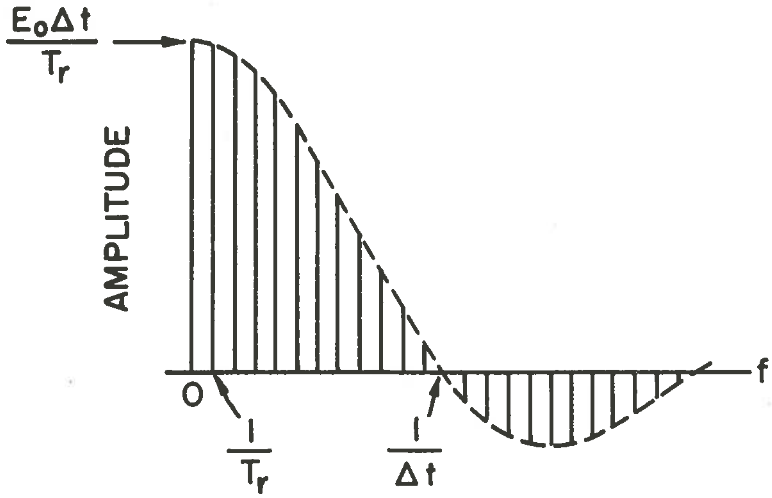
This is shown in Fig. 1B.

B. Received Wave Spectrum

The transmitted pulses may be reflected back to the antenna by some object, arriving at a later time, given by $t_a = d/V$ [or $t_a = \frac{d(\text{miles})}{1.86} \times 10^{-5}$] seconds, where t_a represents the time delay between the transmission of a pulse and its reception, d = distance to reflecting object, and V = velocity of the radio waves. However, the frequency spectrum of the received wave will be different if reflected from a moving object, than if reflected from a fixed object.

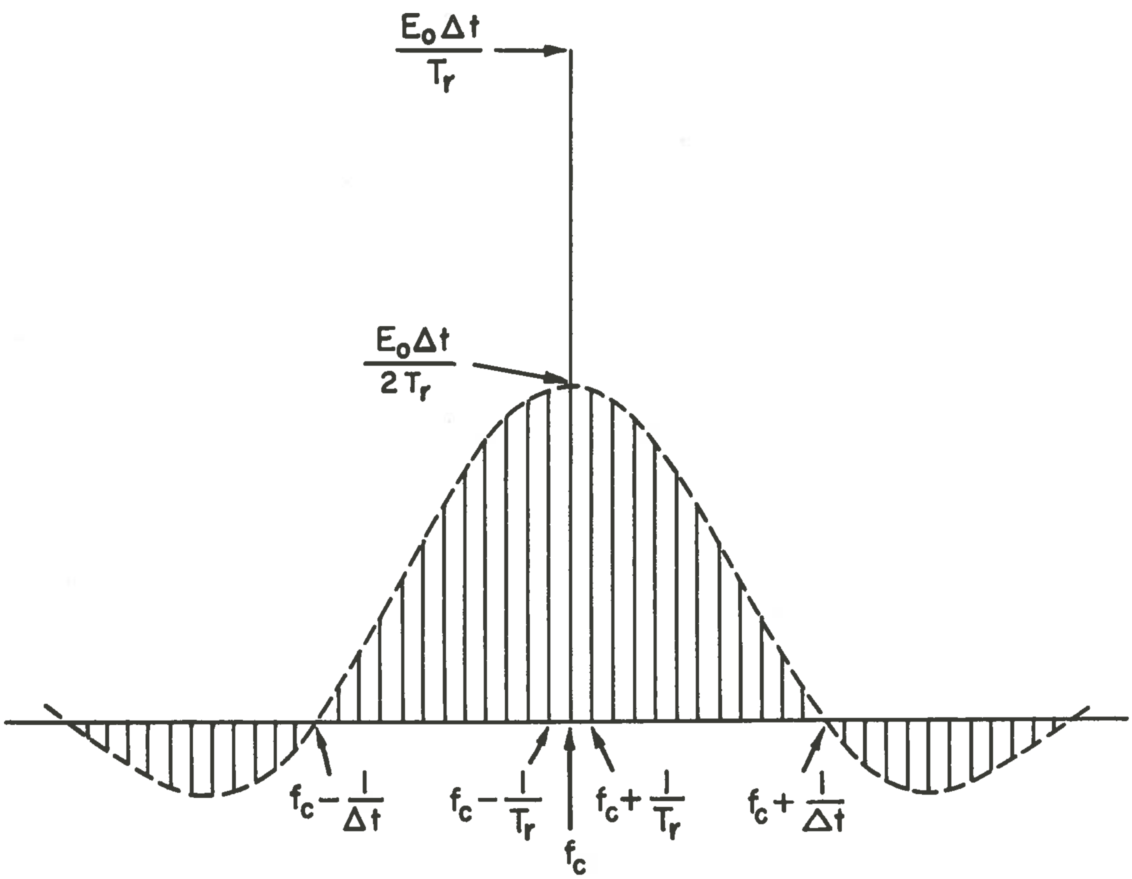
(1) Received Wave from a Fixed Object

The wave received from a fixed object will be the same as the transmitted wave, with the amplitude multiplied by a factor k [which depends on the antenna gain, range, and reflection coefficient of the object].



FREQUENCY SPECTRUM OF MODULATING SIGNAL

FIG. 1A



FREQUENCY SPECTRUM OF TRANSMITTED SIGNAL

FIG. 1B

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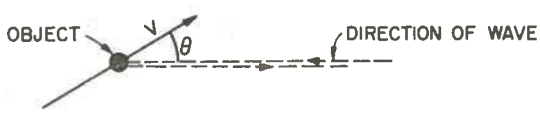
The difference in time may be represented by a phase lag in all of the frequencies, but this will be ignored for the time being. [To take account of this difference in time, a phase angle will be introduced later, when demodulation is considered.]

Thus the spectrum of the received wave will be given by:

$$E_R = kE_o \left[\frac{\Delta t}{T_r} \cos \omega_c t + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos(\omega_c + \frac{2n\pi}{T_r})t + \cos(\omega_c - \frac{2n\pi}{T_r})t \right) \right] \qquad (1.3)$$

(2) Received Wave from a Moving Object

Consider an incident wave of frequency f , on a reflecting object moving with velocity V at an angle of θ to the direction of propagation of the wave [see Fig. 2].



REFLECTION FROM A MOVING OBJECT
FIG. 2

The reflected wave will then be shifted in frequency, owing to the Doppler effect, by an amount $\frac{2V\cos\theta}{\lambda}$, $(0 < \theta < 2\pi)$. This will happen to each of the frequencies present in the transmitted pulse train.

Let $\frac{4\pi V\cos\theta}{\lambda} = C_1$ [As all of the significant frequencies in this case are close to f_c , the carrier frequency, the variation in λ can be considered negligible.]

Then, for a moving object,

$$E_R = kE_o \left[\frac{\Delta t}{T_r} \cos(\omega_c + C_1)t + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos(\omega_c + \frac{2n\pi}{T_r} + C_1)t \right.\right.$$

$$\left.\left. + \cos(\omega_c - \frac{2n\pi}{T_r} + C_1)t\right)\right] . \qquad (1.4)$$

C. Demodulation

In the receiver, the received signal is first mixed and converted to an intermediate frequency. This does not change the received signal significantly; actually it only replaces f_c with f_{if} . It is then passed through a band-pass intermediate frequency filter, which has the effect of eliminating all frequency components above $f_{if} + n/T_r$ and below $f_{if} - n/T_r$.

Thus the received signals now will be

For a fixed object:

$$E'_r = k'E_o \left[\frac{\Delta t}{T_r} \cos \omega_{if}t + \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos(\omega_{if} + \frac{2n\pi}{T_r})t \right.\right.$$

$$\left.\left. + \cos(\omega_{if} - \frac{2n\pi}{T_r})t \right)\right] \qquad (1.5)$$

and for a moving object:

$$E'_r = k'E_o \left[\frac{\Delta t}{T_r} \cos(\omega_{if} + C_1)t + \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos(\omega_{if} + \frac{2n\pi}{T_r} + C_1)t \right.\right.$$

$$\left.\left. + \cos(\omega_{if} - \frac{2n\pi}{T_r} + C_1)t \right)\right] . \qquad (1.6)$$

[k' is now a constant including k , mixer, and i-f gains].

Final demodulation is accomplished by a synchronous detector, which multiplies the intermediate-frequency signal by a voltage at intermediate frequency. This multiplying frequency is obtained from the same master oscillator from which the transmitted signal and the "local oscillator" signal for the mixer is obtained. Thus the whole system is coherent in phase.

Depending on the range to the reflecting object [i.e., path length] and various other factors, the multiplying signal will be at some phase angle ϕ to the intermediate-frequency signal. Therefore, let this signal be

$$E_1 \cos\left(\omega_{if}\, t + \phi\right) .$$

1. Demodulation of Reflected Signal from a Fixed Object

The output video signal will be:

$$E_v = k' E_1 E_o \left[\frac{\Delta t}{T_r} \cos \omega_{if} t \cos(\omega_{if} t + \phi) + \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T}}{n} \left(\cos(\omega_{if} + \frac{2n\pi}{T_r}) t \right.\right. \tag{1.7}$$

$$\left.\left. \cos(\omega_{if} t + \phi) + \cos(\omega_{if} - \frac{2n\pi}{T_r}) t \cos(\omega_{if} t + \phi) \right) \right] .$$

Reducing by trigonometry:

$$E_v = k' E_1 E_o \frac{\Delta t}{2T_r} \left[\cos(2\omega_{if} t + \phi) + \cos \phi \right]$$

$$+ \frac{k' E_1 E_o}{2\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left\{ \cos \left[(2\omega_{if} + \frac{2n\pi}{T}) t + \phi \right] \right.$$

$$\left. + \cos \left[\frac{2n\pi}{T} t - \phi \right] + \cos \left[(2\omega_{if} - \frac{2n\pi}{T}) t + \phi \right] + \cos \left[\frac{2n\pi}{T} t + \phi \right] \right\} . \tag{1.8}$$

After passing through a low-pass filter, which will remove the terms having frequencies near $2f_{if}$, the video signal will be:

$$E_v = k'E_1E_o \cos\phi \left[\frac{\Delta t}{2T_r} + \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{t}{T_r}}{n} \cos \frac{2n\pi}{T_r} t \right]. \quad (1.9)$$

Providing N is high enough, this will appear as a series of pulses, delayed in time from the transmitted pulses by an amount depending on the path length, and of amplitude $\frac{k'}{2} E_1E_o \cos\phi$. It may be noted here that the value of the amplitude will vary with the phase angle ϕ, and actually, for certain values of ϕ, it will be zero. It may also be noted that the amplitude may be either positive or negative, again depending on ϕ.

2. Demodulation of Reflected Signal from a Moving Object

After detection, the video signal will be

$$E_v = k'E_1E_o \Bigg[\frac{\Delta t}{T_r} \cos(\omega_{if} + C_1)t \cos(\omega_{if}t + \phi)$$

$$+ \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos(\omega_{if} + \frac{2n\pi}{T_r} + C_1)t \cos(\omega_{if}t + \phi) \right.$$

$$\left. + \cos(\omega_{if} - \frac{2n\pi}{T_r} + C_1)t \cos(\omega_{if}t + \phi) \right) \Bigg]. \quad (1.10)$$

Reducing to component frequencies by trigonometry,

$$E_v = \frac{k'E_1E_o\Delta t}{2T_r} \left[\cos\left((2\omega_{if} + C_1)t + \phi\right) + \cos(C_1 t - \phi) \right]$$

$$+ \frac{k'E_1E_o}{2\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \Bigg[\cos\left((2\omega_{if} + \frac{2n\pi}{T_r} + C_1)t + \phi\right)$$

$$+ \cos\left((\frac{2n\pi}{T_r} + C_1)t - \phi\right) + \cos\left((2\omega_{if} \frac{2n\pi}{T_r} + C_1)t + \phi\right)$$

$$+ \cos\left((-\frac{2n\pi}{T} + C_1)\ t - \phi\right) \Bigg]. \quad (1.11)$$

Again, after passing through a low-pass filter to remove frequencies of the order of $2f_{if}$,

$$E_v = k'E_1E_o\left\{\frac{\Delta t}{2T_r}\cos(C_1t - \phi) + \frac{1}{2\pi}\sum_{n=1}^{N}\frac{\sin n\pi\frac{\Delta t}{T_r}}{n}\left[\cos\left(\left(\frac{2n\pi}{T_r} + C_1\right)t - \phi\right)\right.\right.$$

$$\left.\left. + \cos\left(\left(\frac{2n\pi}{T_r} - C_1\right)t + \phi\right)\right]\right\}. \qquad (1.12)$$

In this case it can be seen that each of the original frequencies in the modulating pulse train has been split into an upper and lower sideband, of frequency $n/T_r + C_1/2\pi$ and $n/T_r - C_1/2\pi$, respectively. [The former "d-c" term is shifted to a frequency of $C_1/2\pi$]. In the time domain [again providing N is not too small] this corresponds to a train of pulses, of width Δt, spaced at intervals of T_r, and of amplitude $\frac{k'E_1E_o}{2}\cos(C_1nT_r - \phi)$ [where n here represents the number of pulses which have occurred].

D. The Video Spectra

Fig. 3 shows the frequency spectra of the received waves from a fixed object and from a moving object, with the effect of various arbitrary phase angles ϕ indicated.

It can be seen from the frequency spectra of the video pulses that there is considerably more information available here than from an incoherent radar system. When presented on an A-scan, the result will be fixed amplitude echoes, having either positive or negative polarity for a fixed object. A moving object however, will have a "fluttering" echo, showing both polarities. Because of this, these echoes will now be referred to as bipolar pulses.

The information that can be extracted from the signal in the coherent system is:

1. The range of a reflecting object.
2. Indication of motion of a reflecting object.
3. The radial velocity of a moving object.
4. The radial direction of motion [i.e., whether approaching or receding].

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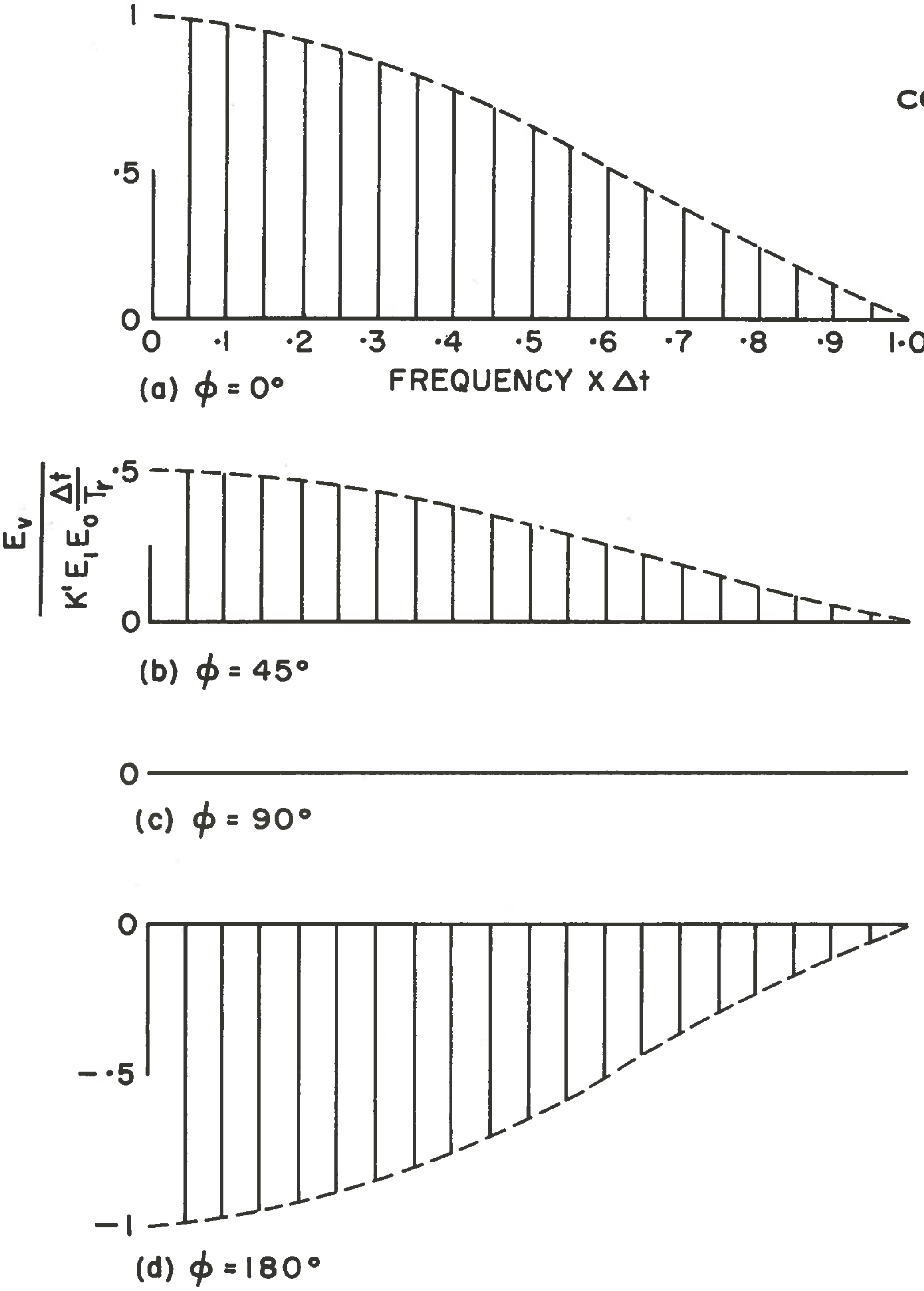
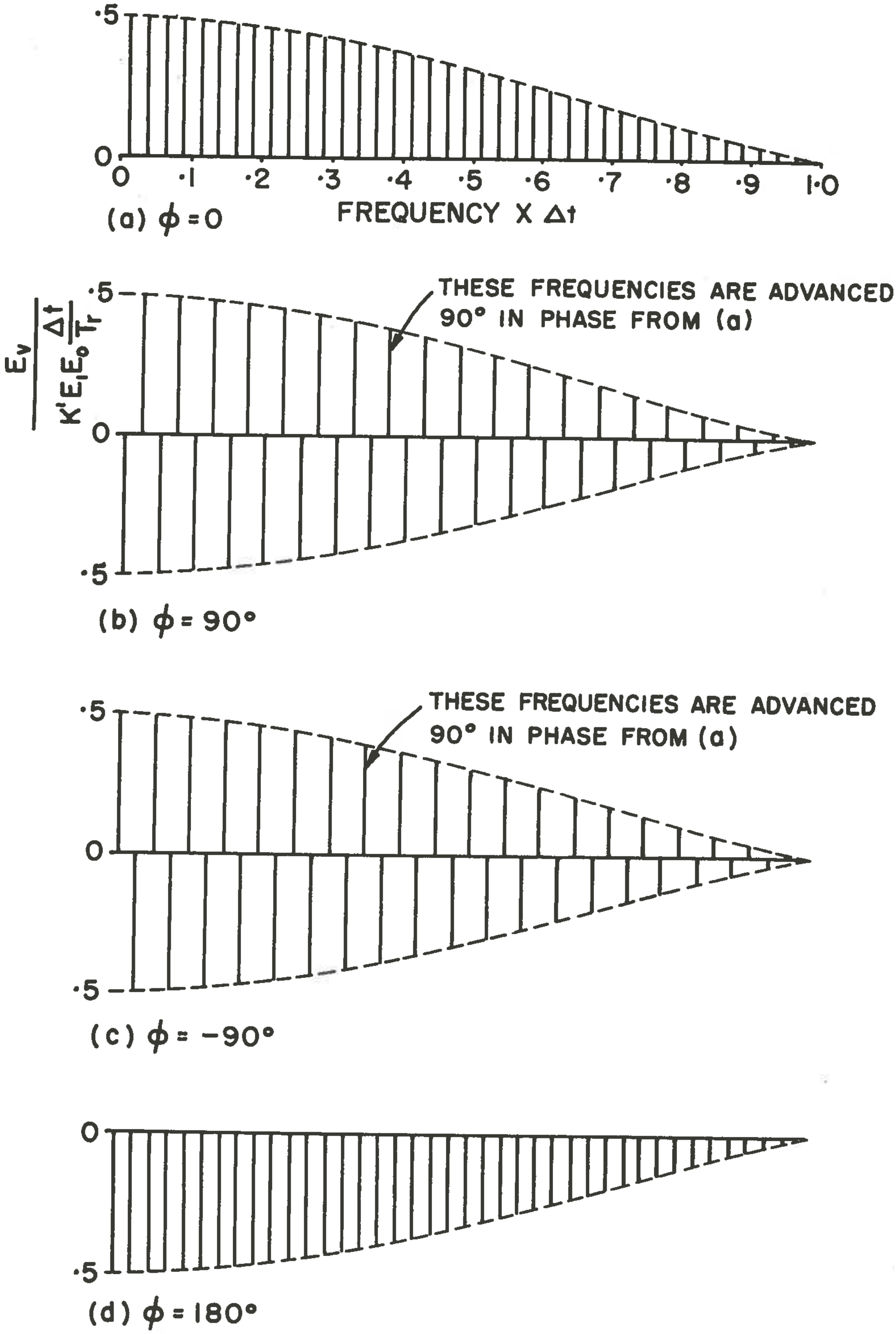


FREQUENCY SPECTRUM (VIDEO) FROM "FIXED" ECHO

$$\frac{\Delta t_1}{T_r} = \cdot 05$$

FIG. 3A



FREQUENCY SPECTRUM (VIDEO) FROM "MOVING" ECHO

$\frac{\Delta t}{T_r} = \cdot 05$; $C = \frac{\pi}{2T_r}$ (approaching)

FIG.3B

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The range can be deduced from the usual techniques of timing the arrival of an echo.

The problem of detection of motion and finding the radial velocity, however, requires the application of different techniques. Part II of this paper will discuss an integrating system which will give the range to an object with a reduced degree of accuracy, but will give an indication of motion.

It is possible to determine radial velocity by using a series of narrow band-pass filters about one of the frequencies present in the spectrum from a fixed object. If there are a number of filters of bandwidth Δf, covering the frequency band n/T_r to $n \pm \frac{1}{2}/T_r$, the value of the Doppler frequency shift $C_1/2\pi$ can be determined, to an accuracy of Δf, by noting the particular filter passing a signal. Obviously, in this case, n should be fairly small, so that the frequency $n/T_r \pm C_1/2\pi$ will be present with a greater amplitude.

The sense of the radial motion [i.e., the sign of C_1] can be distinguished by using two coherent detectors, one fed by a demodulating signal $E_1 \cos(\omega_{if}t + \phi)$, and the other by a signal $E_1\cos(\omega_{if}t + \phi - \pi/2)$ or $E_1 \sin(\omega_{if}t + \phi)$. If the output of the first detector is then advanced by $\pi/2$ and the two signals compared, it will be found that the sign of C_1 has an effect on the result. [See Appendix A for a discussion of this effect.]

PART II: THE THIRD DETECTOR

The total range covered by the system is divided into eight overlapping gates [i.e., after the transmitter pulse, the time required for an echo to return from the maximum range is divided into eight overlapping intervals]. A received echo is then passed through a gate corresponding to its range. As the maximum range discrimination required is given by the width of the gates, all pulses coming through any particular gate can be integrated. Consideration will now be given to the happenings in a particular gate.

A. Operation of the 3rd Detector (Idealized)

(a) Echo from a fixed object.

If a series of pulses of fixed amplitude E, width Δt, and repetition interval T_r, are integrated, the results are:

$$\left.\begin{array}{lll} \text{If} & e = E & mT_r < t < mT_r + \Delta t \\ & e = 0 & mT_r + \Delta t < t < (m+1)T_r \end{array}\right\} m = 1,2,3.........$$

Then on integration:

$$\begin{array}{ll} e = Et + (m-1)E\Delta t & mT_r < t < mT_r + \Delta t \\ e = mE\Delta t & mT_r + \Delta t < t < (m+1)T_r \end{array} \qquad (2.1)$$

This corresponds to a "staircase" type of function, which will increase continuously. [However, as will be seen later, the practical circuit is not a perfect integrator, and thus the voltage does not increase indefinitely.]

If the frequency spectrum of an echo from a fixed object is integrated, the result will be the "frequency" spectrum of the aforementioned "staircase" function. This integration results in

$$E_{out} = k'E_1E_o \cos\phi \left[\frac{\Delta t}{2T_r}t + \frac{T_r}{2\pi^2}\sum_{n=1}^{N}\frac{\sin n\pi\frac{\Delta t}{T_r}}{n^2}\sin\frac{2n\pi}{T_r}t\right] . \qquad (2.2)$$

As can be seen, this is a rather odd "frequency" function, as the amplitude of the "d-c" term varies with time. However, the significant part is that the "sin kx/x" term representing the amplitudes of the frequencies other than "d-c" is now a $\sin kx/x^2$ type of term. Thus the energy of the pulse has been concentrated more in the lower frequencies, and therefore the effective bandwidth of the system has been considerably narrowed.

(b) Echo from a moving object.

The amplitude of the m^{th} pulse from a moving object is $E \cos C_1(mT_r)$ assuming a shift in the time axis to eliminate phase angles. As the phase angle here is arbitrary, it can be assumed zero without loss of generality.

$$\text{Then:}\quad \left.\begin{aligned} e &= E\cos C_1(m-1)T_t \qquad & mT_r < t < mT_r + \Delta t \\ e &= 0 & mT_r + \Delta t < t < (m+1)T_r \end{aligned}\right\} m = 1,2,3\ldots\ldots$$

On integrating, considering all pulses up to and including the M^{th},

$$\begin{aligned} e = V_M &= Et\cos C_1(m-1)T_r + E\Delta t \sum_{m=1}^{M-1} \cos C_1(m-1)T_r \qquad & MT_r < t < MT_r + \Delta t \\ &= E\Delta t \sum_{m=1}^{M} \cos C_1(m-1)T_r & MT_r + \Delta t < t < (M+1)T_r \end{aligned} \tag{2.3}$$

This expression for e between the M^{th} and $(M+1)^{th}$ pulses can be reduced further, as follows:

$$\begin{aligned} V_M &= E\Delta t \sum_{m=1}^{M} \cos C_1(m-1)T_r = E\Delta t \sum_{m=1}^{M} \mathrm{Re}\left[\varepsilon^{j(m-1)C_1T_r}\right] \\ &= E\Delta t\ \mathrm{Re} \sum_{m=1}^{M} \varepsilon^{j(m-1)C_1T_r} \quad \text{(where Re means "real part of").} \end{aligned} \tag{2.4}$$

The series here is a geometric series with $r = \varepsilon^{jC_1T_r}$.

Thus the sum is $\dfrac{1 - \varepsilon^{jMC_1T_r}}{1 - \varepsilon^{jC_1T_r}}$.

$$\text{Then:}\quad V_M = E\Delta t\ \mathrm{Re}\left[\frac{1 - \varepsilon^{jMC_1T_r}}{1 - \varepsilon^{jC_1T_r}}\right],$$

$$= E\Delta t\ \mathrm{Re}\ \frac{1 - (\cos MC_1T_r + j\sin MC_1T_r)}{1 - (\cos C_1T_r + j\sin C_1T_r)}. \tag{2.5}$$

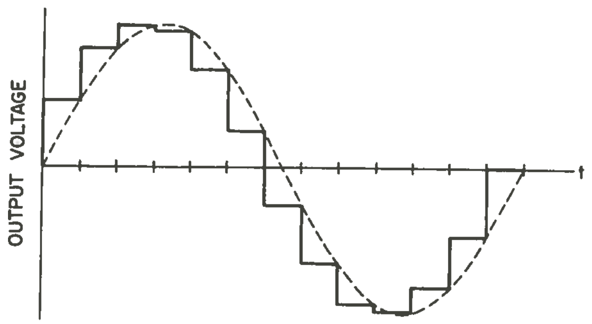
Rationalizing and taking the real part:

$$V_M = \frac{E\Delta t}{2}\left[1 + \frac{\cos(M-1)C_1T_r - \cos MC_1T_r}{1 - \cos C_1T_r}\right] .$$

As $\cos C_1T_r$ will be a constant for any given values of C_1 and T_r, let the factor $1 - \cos C_1T_r = D$.

Then:
$$V_M = \frac{E\Delta t}{2}\left[1 + \frac{\cos(M-1)C_1T_r - \cos MC_1T_r}{D}\right] . \qquad (2.6)$$

This is a series of "boxcars", or constant levels of voltage between pulses with rapid jumps from level to level during the pulse, the whole having the general shape of a sine wave [see Fig. 4]. [It is assumed that the pulse width is narrow enough compared with T_r, so that the nature of the jump between the constant levels is unimportant.]



OUTPUT OF IDEAL INTEGRATOR FOR MOVING OBJECT

$\frac{2\pi}{C_1} = 13\ T_r$

FIG. 4

Again, integration of the frequency spectrum of the bipolar pulse train will result in the frequency spectrum of these "boxcars". This is given by:

$$E_{out} = k'E_1E_o \left\{ \frac{\Delta t}{2CT_r} \sin(C_1 t - \phi) \right.$$

$$\left. + \frac{1}{2\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\frac{\sin\left[(\frac{2n\pi}{T_r} + C_1)t - \phi\right]}{\frac{2n\pi}{T_r} + C_1} + \frac{\sin\left[(\frac{2n\pi}{T_r} - C_1)t + \phi\right]}{\frac{2n\pi}{T_r} - C_1} \right) \right\}$$

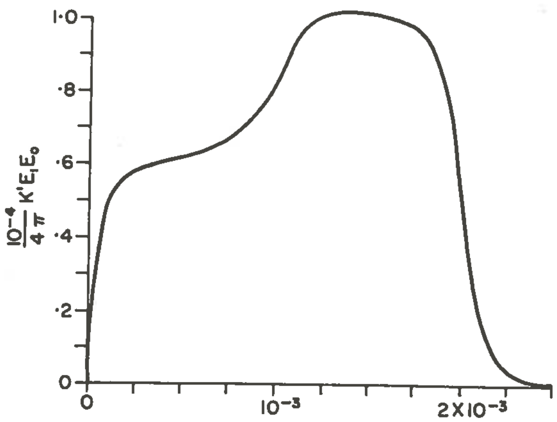
or

$$E_{out} = k'E_1E_o \left\{ \frac{\Delta t}{2CT_r} \sin(C_1 t - \phi) \right.$$

$$\left. + \frac{T_r}{4\pi^2} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n^2} \left(\frac{\sin\left[(\frac{2n\pi}{T_r} + C_1)t - \phi\right]}{1 + \frac{C_1T_r}{2\pi n}} + \frac{\sin\left[(\frac{2n\pi}{T_r} - C_1)t + \phi\right]}{1 - \frac{C_1T_r}{2\pi n}} \right) \right\} \quad (2.7)$$

It will be noted that again the $\sin kx/x$ representing the amplitudes is now $\sin kx/x^2$, and the net result is a concentration of the energy in the lower frequencies. [The effect of the $1 \pm C_1T_r/2\pi n$ term decreases rapidly as n increases.] Also, here, it will be noted the "fundamental" frequency $C_1/2\pi$ has an amplitude that decreases with $1/C_1$.

The effectiveness of the concentration of energy in the low frequencies is shown by the accompanying plot of half a cycle of the Doppler frequency, using frequencies up to, and including, $n = 4$ [see Fig. 5].



PLOT OF "BOXCAR" OUTPUT FROM IDEAL INTEGRATOR

$C_1 = 400\pi$ (Doppler frequency $f_d = 200$ cy.)

$T_r = \cdot 001$ sec.

$\Delta t = 2 \times 10^{-5}$ sec.

$\phi = 0$

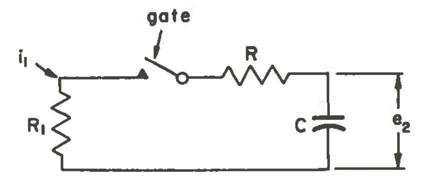
for "n" up to 4

FIG. 5

B. Operation of the Third Detector (Practical)

The third detector actually consists of a balanced 4-diode switch, which is gated at an appropriate time after the transmitter pulse for the range of the particular channel under consideration. This feeds into an integrating circuit connected to the grid of an amplifier stage. The voltage existing on the capacitor of the integrating circuit is held between gates at the level existing at the end of the preceding gate [neglecting leakage].

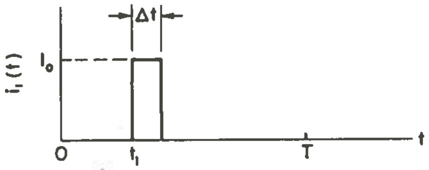
The equivalent circuit of the switch and integrator can be assumed as shown [see Fig. 6] .



EQUIVALENT CIRCUIT OF 3RD DETECTOR

FIG. 6

As the opening of the "switch" leaves a certain voltage on the capacitor, C , which remains until the switch closes again, the results can be arrived at by considering only the time the "switch" is closed. As a start, consider a completely discharged circuit, to which an echo pulse is applied t_1 seconds after the gate opens [i.e., after the "switch" closes]. The pulse is Δt seconds wide, and the gate is open for T seconds [see Fig. 7].



SIGNAL APPLIED TO 3RD DETECTOR

FIG. 7

Then, $$\frac{e_2(T)}{I_oR_1} = \varepsilon^{-k_1(1 - \frac{t_1}{T})}\left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1\right] \qquad T > t_1 + \Delta t$$

$$= \varepsilon^{-k_1}\left[\varepsilon^{k_1\frac{\Delta t}{T}} - 1\right], \qquad t_1 < T < t_1 + \Delta t \qquad (2.8)$$

where $k_1/T = 1/C(R_1 + R)$.

[See Appendix B for derivation of this expression].

Figs. 8A and 8B show the resultant voltage left on the capacitor as a function of the position of the pulse in the gate. This is shown for two pulse widths, and for three values of k_1 for each pulse width. It can be seen from these plots that if the value of k_1 is not carefully chosen, the range response of the system [within each gate] will be quite poor.

To fix the value of k_1 that will result in a maximum response for a pulse occurring at the start of the gate [i.e., $t_1 = 0$] differentiate the expression $e_2(T)/I_oR_1$ with respect to k_1.

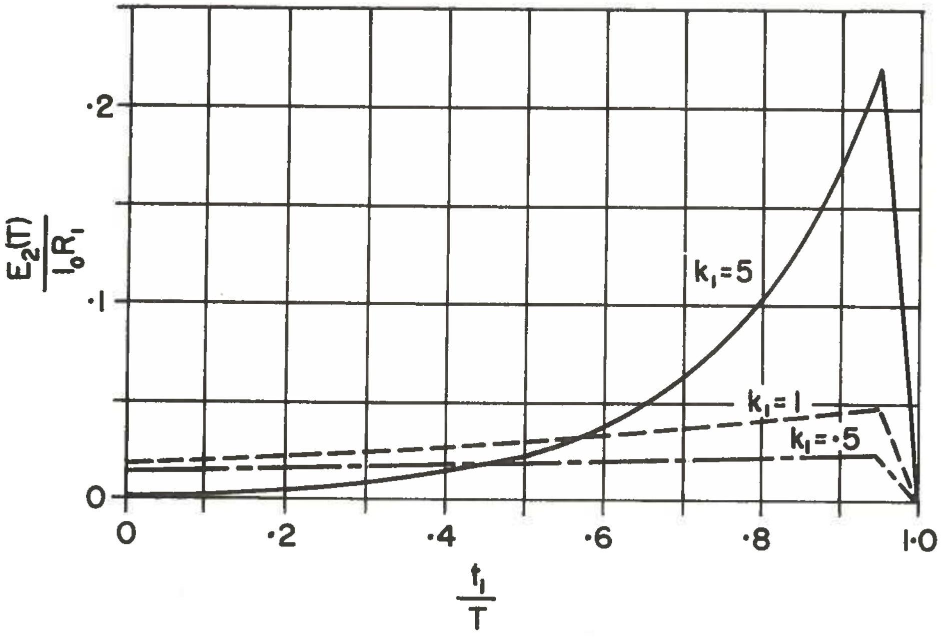
If $$\frac{e_2(T)}{I_oR_1} = \varepsilon^{-k_1}\left[\varepsilon^{k_1\frac{\Delta t}{T}} - 1\right] \qquad (\text{for } t_1 = 0),$$

then
$$\frac{d}{dk_1}\left[\frac{e_2(T)}{I_oR_1}\right] = -\varepsilon^{-k_1}\left[\varepsilon^{k_1\frac{\Delta t}{T}} - 1\right] + \varepsilon^{-k_1}\frac{\Delta t}{T}\varepsilon^{k_1\frac{\Delta t}{T}} = 0 \text{ for maximum response.}$$

Then, $$\varepsilon^{k_1\frac{\Delta t}{T}}\left(1 - \frac{\Delta t}{T}\right) = 1,$$

or $$k_1 = \frac{T}{\Delta t}\ln\frac{1}{1 - \frac{\Delta t}{T}}. \qquad (2.9)$$

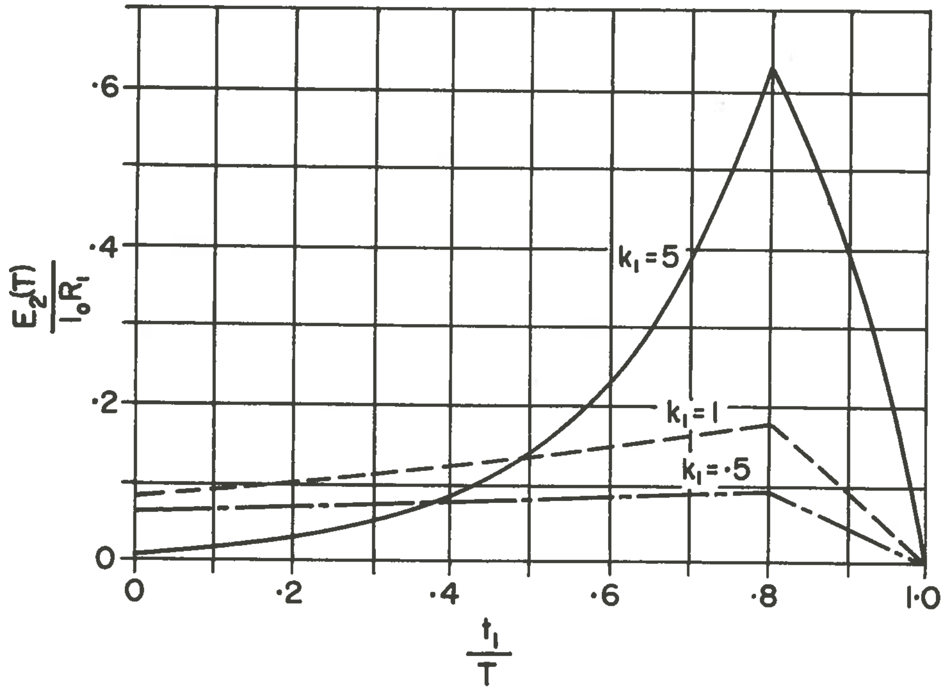
This means that, for flattest response, the integrating time constant $C(R_1 + R)$ should equal $\dfrac{\Delta t}{\ln\frac{1}{1 - \frac{\Delta t}{T}}}$.



OUTPUT OF 3RD DETECTOR FOR SINGLE PULSE

($\Delta t = \cdot 05T$)

FIG. 8A



OUTPUT OF 3RD DETECTOR FOR SINGLE PULSE

($\Delta t = \cdot 2T$)

FIG. 8B

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Various compensating devices [usually inductances] have been tried to eliminate the variation of amplitude with position of the pulse in the gate, and have been found more or less successful. It is not proposed, however, to analyze the effect of these at this time.

Having obtained the result of a single pulse in a gate, the effect of a series of pulses, having the same position in the gate, but occurring in a succession of gates will be considered.

If the capacitor, C, is at an initial voltage V_o, it will discharge during the time the gate is open. At the end of the gate its voltage is given by

$$V_c = V_o \, \varepsilon^{-\frac{T}{C(R_1 + R)}} = V_o \, \varepsilon^{-k_1} .$$

By superposition, this can be added to the capacitor voltage arising from a pulse within the gate.

$$\text{Then,} \quad e_2(T) = I_o R_1 \, \varepsilon^{-k_1(1 - \frac{t_1}{T})} \left[\varepsilon^{\frac{k_1 \Delta t}{T}} - 1 \right] + V_o \, \varepsilon^{-k_1} . \tag{2.10}$$

(a) <u>Effect of Successive Pulses of the Same Amplitude</u> (i.e., pulses from a "fixed" echo).

The pulses will recur at a repetition period of T_r, which is also equal to the repetition period of the gate openings. Therefore they will occur in the same positions in the gate.

Upon the commencement of an echo pulse:

At the end of the first gate opening -

$$V_1 = I_o R_1 \, \varepsilon^{-k_1(1 - k_1 \frac{t_1}{T})} \left[\varepsilon^{\frac{k_1 \Delta t}{T}} - 1 \right] .$$

This is equal to the "initial" condenser voltage, "V_o", at the start of the second opening.

Then, at end of 2nd gate opening,

$$V_2 = I_oR_1 \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})}\left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1\right] + I_oR_1 \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})}\left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1\right]\varepsilon^{-k_1},$$

or

$$V_2 = I_oR_1 \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})}\left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1\right]\left[1 + \varepsilon^{-k_1}\right]. \qquad (2.11)$$

Similarly, at end of the 3rd gate opening,

$$V_3 = [I_oR_1] \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})}\left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1\right]\left[1 + \varepsilon^{-k_1} + \varepsilon^{-2k_1}\right], \qquad (2.12)$$

or, at the end of the m'th gate opening,

$$V_m = K\left[1 + \varepsilon^{-k_1} + \varepsilon^{-2k_1} + \ldots\ldots\ldots\ldots + \varepsilon^{-(m - 1)k_1}\right], \qquad (2.13)$$

where $K = I_oR_1 \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})}\left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1\right]$.

As the series $1 + \varepsilon^{-k_1} + \varepsilon^{-2k_1} \ldots\ldots\ldots\ldots + \varepsilon^{-(m - 1)k_1}$ is a geometric series of ratio ε^{-k_1}, its sum is $\frac{1 - \varepsilon^{-mk_1}}{1 - \varepsilon^{-k_1}}$, or as $m \to \infty$,

$$V_m = \frac{K}{1 - \varepsilon^{-k_1}}. \qquad (2.14)$$

It may be noted that this result is considerably different from that obtained with a "perfect integrator". Instead of a voltage level increasing constantly in a "staircase" manner, each step is of a smaller amplitude, and, in the limit, the level becomes $k/1-\varepsilon^{-k_1}$. Thus a fixed echo results only in a change in the d-c voltage level on the capacitor [as applied to the grid of the following amplifier stage.] *

* Note: The above discussion, as in all of the discussions on this circuit, assumes that what actually happens during the opening of the gate is of a short enough duration, in comparison with the pulse (and gate) repetition period, that its effect on the analysis is insignificant — the only significant item being the voltage level between the gate openings.

(b) Effect of Bipolar Pulses

In this case, the input current [i.e., input pulse amplitude] can be given by

$$I_o = I_{om} \cos [(m - 1)C_1T_r] \qquad (m = 1,2,3,\ldots\ldots)$$

where m again is the number of pulses which have occurred. [The assumption of the cosine function without arbitrary phase angle does not result in any loss of generality here. The factor (m - 1) takes into account the first pulse, as cos 0 = 1.]

Then, at end of first gate opening:

$$V_1 = I_{om}R_1 \, \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{\frac{k_1 \Delta t}{T}} - 1 \right].$$

As this is the voltage on the condenser at the start of the second gate opening:

$$V_2 = I_{om} \cos (C_1T_r)R_1 \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{\frac{k_1 \Delta t}{T}} - 1 \right]$$

$$+ I_{om} R_1 \, \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{\frac{k_1 \Delta t}{T}} - 1 \right] \varepsilon^{-k_1}$$

$$= I_{om}R_1 \, \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{k_1\frac{\Delta t}{T}} - 1 \right] \left[\cos C_1T_r + \varepsilon^{-k_1} \right]. \qquad (2.15)$$

Similarly, at end of second gate opening

$$V_3 = I_{om}R \, \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{k_1\frac{\Delta t}{T}} - 1 \right] \left[\cos 2C_1T_r + \varepsilon^{-k_1} \cos C_1T_r + \varepsilon^{-2k_1} \right]$$

$$= K \left[\cos 2C_1T_r + \varepsilon^{-k_1} \cos C_1T_r + \varepsilon^{-2k_1} \right], \qquad (2.16)$$

where K here is $I_{om}R_1 \, \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{k_1\frac{\Delta t}{T}} - 1 \right]$.

Continuing, after the M^{th} gate opening,

$$V_M = K \sum_{m=1}^{M} \cos(m-1)\, C_1 T_r \, \varepsilon^{-(M-m)k_1} ,$$

$$\text{or} \quad V_M = K\, \varepsilon^{-MK_1} \sum_{m=1}^{M} \cos(m-1)\, x\, \varepsilon^{mk_1} , \qquad (\text{where } x = C_1 T_r). \quad (2.17)$$

This can be expressed as

$$V_M = K\, \varepsilon^{-Mk_1} \sum_{m=1}^{M} \operatorname{Re}\left[\varepsilon^{mk_1 + j(m-1)x}\right]$$

$$= K^{-(M-1)k_1} \operatorname{Re}\left[\sum_{m=1}^{M} \varepsilon^{(m-1)(k_1 + jx)}\right]. \qquad (2.18)$$

The sum here is a geometric series with $r = \varepsilon^{k_1 + jx}$, and $n = m - 1$.

$$\text{Then} \quad V_M = K\, \varepsilon^{-(M-1)\,k_1} \operatorname{Re}\left[\frac{1 - \varepsilon^{M(k_1 + jx)}}{1 - \varepsilon^{k_1 + jx}}\right]. \qquad (2.19)$$

Substituting $\cos x + j \sin x$ for ε^{jx} and evaluating the real part of the result gives

$$V_M = K \frac{\varepsilon^{-(M-1)k_1} - \varepsilon^{-(M-2)k_1} \cos C_1 T_r - \varepsilon^{k_1} \cos MC_1 T_r + \varepsilon^{2k_1} \cos(M-1)\, C_1 T_r}{1 - 2\, \varepsilon^{k_1} \cos C_1 T_r + \varepsilon^{2k_1}} . \qquad (2.20)$$

The first two terms in the numerator represent a transient which, as M becomes greater [i.e., more pulses have occurred] will die out. Thus, when M is large, the "steady state" response given below will be obtained.

$$V_M = K\, \varepsilon^{k_1} \frac{\varepsilon^{k_1} \cos(M-1) C_1 T_r - \cos MC_1 T_r}{1 - 2\, \varepsilon^{k_1} \cos C_1 T_r + \varepsilon^{2k_1}} . \qquad (2.21)$$

Here, the part of the expression $\dfrac{K\,\varepsilon^{k_1}}{1 - 2\,\varepsilon^{k_1} \cos C_1T_r + \varepsilon^{2k_1}}$ will be independent of the number of pulses M which have occurred.

Let this quantity be K'.

Then
$$V_M = K'\left[\varepsilon^{k_1} \cos(M - 1)C_1T_r - \cos MC_1T_r\right]. \qquad (2.22)$$

This may be compared directly with Equation 2.6, the expression for the ideal integrator,

$$V_M = \frac{E\Delta t}{2}\left[1 + \frac{\cos(M - 1)C_1T_r - \cos MC_1T_r}{D}\right].$$

The "d-c" term in the second expression is not completely general, but is rather a result of the arbitrary phase angle of zero used in the derivation. Depending on what phase angle is "picked", this "constant" will vary between plus and minus one.

Thus it would appear that the significant difference between the two expressions [apart from the constants] is the factor ε^{k_1} present in the expression for the practical circuit. The degree of difference of behaviour of the practical circuit from the ideal integrator can then be indicated by the value of k_1. For example,

$$\text{As } k_1 \to 0, \qquad \varepsilon^{k_1} \to 1.$$

Then
$$V_M \to K'\left[\cos(M - 1)C_1T_r - \cos MC_1T_r\right]$$

$$\to \frac{K}{2}\,\frac{\cos(M - 1)C_1T_r - \cos MC_1T_r}{1 - \cos C_1T_r}$$

$$\left[\text{as } K \to \frac{K}{2(1 - \cos C_1T_r)} \text{ when } k_1 \to 0\right].$$

Also,

$$\text{since} \quad K = I_oR_1\, \varepsilon^{-k_1(1 - k_1\frac{t_1}{T})} \left[\varepsilon^{\frac{k_1\Delta t}{T}} - 1 \right],$$

as $k_1 \to 0$

$$K \to I_oR_1 \left[1 + \frac{k_1\Delta t}{T} - 1 \right]$$

$$= I_oR_1 \frac{k_1\Delta t}{T} = \frac{I_oR_1}{C(R + R_1)} \Delta t \,.$$

So as $k_1 \to 0$

$$V_M \to \frac{I_oR_1}{2C(R + R_1)} \Delta t \left[\frac{\cos(M - 1)C_1T_r - \cos MC_1T_r}{1 - \cos C_1T_r} \right], \qquad (2.23)$$

or the response of the actual circuit approaches that of the ideal integrator. On the other hand, as k_1 becomes large:

$$V_M \to K'\, \varepsilon^{k_1} \cos(M - 1)C_1T_r$$

$$= K \frac{\varepsilon^{2k_1} \cos(M - 1)C_1T_r}{\varepsilon^{2k_1}} = K \cos(M - 1)C_1T_r \,. \qquad (2.24)$$

Thus it would appear that the relative response is improved considerably with an increase in k_1 . However, this consideration must be balanced with the response as a function of the position of the pulse in the gate, as shown by the curves in Fig. 8. There it is shown that the relative response as a function of position in the gate deteriorates badly as k_1 increases, the optimum value being given by Equation 2.9.

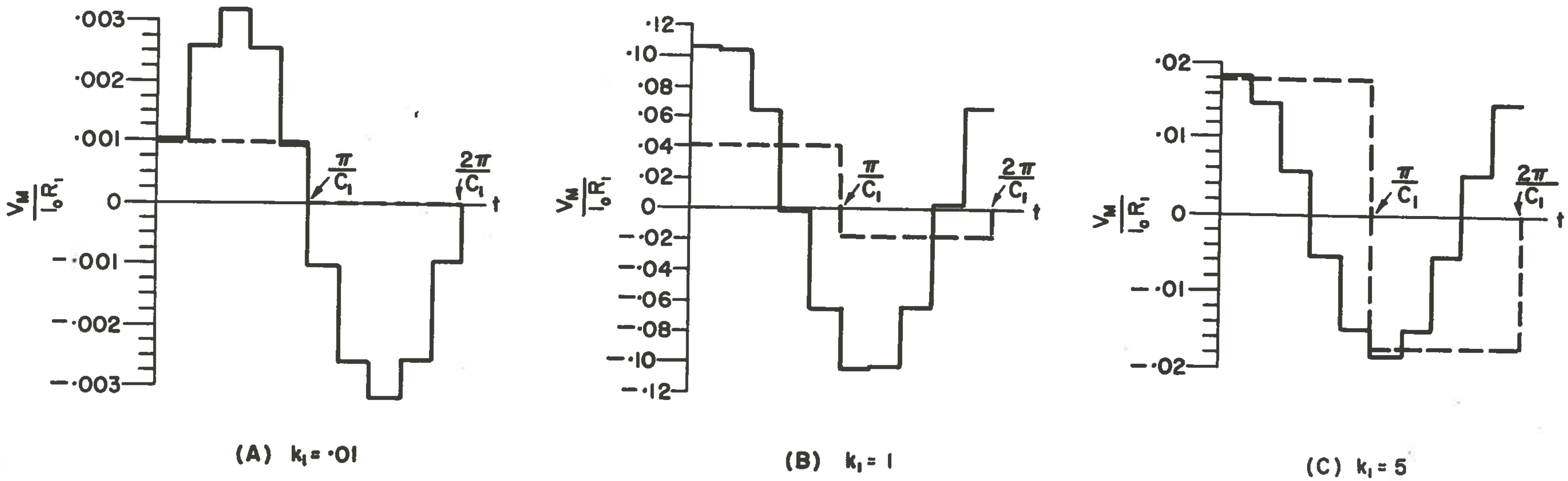
As an illustration, waveforms of the outputs from:

(a) The ideal integrator (actually the practical circuit, but with $k_1 = 0.01$).

(b) Practical circuit, $k_1 = 1$

(c) Practical circuit, $k_1 = 5$

are given for $C_1T_r = \pi$ and $C_1T_r = \pi/5$ [see Figs. 9A,B, and C].



for $\frac{\Delta t}{T} = \cdot 2$; $t_1 = 0$

— — $C_1 T_r = \pi$

—— $C_1 T_r = \frac{\pi}{5}$

OUTPUT OF GATE CIRCUIT

FIG. 9

The practical circuit with $k_1 = 0.01$ is taken, rather than the ideal integrator, so the relative amplitudes of the outputs can be shown. [The difference in the constants between the ideal integrator and the practical circuit would make this difficult otherwise.] Also, for the sake of uniformity, the input pulse is assumed at the beginning of the gate in all cases, and $\Delta t/T$ is taken equal to 0.2.

This shows the effect on output amplitude of having an "imperfect" integrator. As has been shown before [Eq. 2.7], ideal integration results in a serious falling off in output amplitude with increasing Doppler frequency. As the time constant of the integrating circuit is reduced [k_1 increased] this falling off becomes less noticeable, until, with k_1 large, the output amplitude becomes substantially constant with increasing Doppler frequency. Against this result, however, must be balanced the fact that the output amplitude as a function of the position of the pulse in the gate becomes poorer.

(c) Frequency Spectrum of Output

The output spectrum from the ideal integrator has been given in Equations 2.2 and 2.7. It will be noted that this is effectively an extremely narrow low-pass filter. As the time-constant of the integrating circuit is reduced [k_1 increased] the lower frequencies are no longer integrated, and, effectively, the bandwidth of the low-pass filter is increased. This will result in the lower-frequency terms in the frequency expression for E_{in} being passed without modification, and the higher-frequency terms being integrated. The resultant frequency expression, then, would be quite complicated. The result, qualitatively, is that the effect of the Doppler frequency on output amplitude will be eliminated for the terms up to a certain frequency. This effect can be noticed in Fig. 9.

In this system, the maximum Doppler frequency recognizable is $f_d = C_1/2\pi = 1/2T_r$, as for values of f_d above this, say from $f_d = 1/2T_r$ to $f_d = 1/T_r$, the frequency spectrum will be the same as for $f_d' = 1/T_r - f_d$. Therefore, there is no means of distinguishing between f_d and f_d'. Thus it would be desirable to have the variation of output with frequency small at least up to $f_d = 1/2T_r$. This would correspond to $k_1 > \pi T/T_r$, which, for a typical system (T = 100 µs., T_r = .001) would be $k_1 > 0.3$. Such values of k_1 do not introduce a serious difficulty with respect to pulse position in the gate [e.g., the best overall response, for a system where $\Delta t = 0.2T$, is at a value of approximately $k_1 = 1$.]

CONCLUSIONS

It is obvious that the choice of k_1 is the matter of prime concern here, especially as Δt and T are likely to be chosen from other considerations. The factors to be considered in its choice are — frequency response over the system, response with pulse position in the gate, and overall bandwidth desired in the system [as the bandwidth of this filter is effectively the limiting bandwidth in the whole system.] It must be kept in mind that a poor frequency response can be corrected [at least partly] by equalizers after the gates of the 3rd detector, but correction for poor response with variation of pulse position in the gate will be considerably more difficult.

This discussion has been idealized, with no attempt to show or discuss the actual circuits used to bring about the result. However, experimental results using a harmonic analyser verify the statements made about frequency response, and observation on an oscilloscope agrees with the waveforms arrived at by the theoretical method.

APPENDIX A

A METHOD OF INDICATING THE DIRECTION OF MOTION OF AN OBJECT

From Fig. 2, it can be seen that the approach or recession of an object will result in a change of sign of C_1 , where $C_1/2\pi$ is the Doppler shift in frequency. As the value of the arbitrary phase angle ϕ introduced in the detector is not easy to control, the determination of the sign of C_1 should not depend on it. The 2-channel system considered here has the required independence.

Consider two detectors, one in which the multiplying signal is $E_1 \cos(\omega_{if}\ t + \phi)$ and the other in which it is $E_1\ \sin(\omega_{if}\ t + \phi)$ [signals of the same intermediate frequency going into both].

I For C_1 positive (object approaching)

Channel I (multiply by $E_1\ \cos(\omega_{if} + \phi)$)

$$E_{v1} = k_1' E_1 E_o \left\{ \frac{\Delta t}{2T_r} \cos(C_1\ t - \phi) + \frac{1}{2\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\cos \left[(\frac{2n\pi}{T_r} + C_1)t - \phi \right] \right.\right.$$

$$\left.\left. + \cos \left[(\frac{2n\pi}{T_r} - C_1)\ t + \phi \right] \right) \right\} .$$

Channel II (multiply by $E_1\ \sin(\omega_{if}\ t + \phi)$)

After reducing:

$$E_{v2} = k' E_1 E_o \left\{ - \frac{\Delta t}{2T_r} \sin(C_1\ t - \phi) - \frac{1}{2\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\sin \left[(\frac{2n\pi}{T_r} + C_1)t - \phi \right] \right.\right.$$

$$\left.\left. - \sin \left[(\frac{2n\pi}{T_r} - C_1)\ t + \phi \right] \right) \right\} .$$

Now, if E_{v1} is advanced $90°$ in phase by a phase shifter [recalling $\cos(\theta + \pi/2) = -\sin\theta$],

$$E'_{v1} = k'E_1E_o \left\{ -\frac{\Delta t}{2T_r} \sin(C_1 t - \phi) - \frac{1}{2\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\sin\left[(\frac{2n\pi}{T_r} + C_1)t - \phi \right] \right.\right.$$

$$\left.\left. + \sin\left[(\frac{2n\pi}{T_r} - C_1)t + \phi \right] \right) \right\}.$$

Two methods of comparison of the signals E_{v2} and E_{v1} are now indicated.

(1) Simple addition:

$$E_{v2} + E'_{v1} = k'E_1E_o \left\{ \frac{\Delta t}{T_r} \sin(C_1 t - \phi) - \frac{1}{\pi} \sum_{n=1}^{N} \frac{\sin n\pi \frac{\Delta t}{T_r}}{n} \left(\sin\left[(\frac{2n\pi}{T_r} + C_1)t - \phi \right] \right) \right\}.$$

(2) Passing through a low-pass filter, to eliminate all of the higher frequencies (from n = 1 up) and then multiplying:

After the filter, $E_{v2} = -k'E_1E_o \dfrac{\Delta t}{2T_r} \sin(C_1 t - \phi)$

$$E'_{v1} = -k'E_1E_o \frac{\Delta t}{2T_r} \sin(C_1 t - \phi) .$$

Then, $E_{v2}E'_{v1} = (k'E_1E_o \dfrac{\Delta t}{2T_r})^2 \sin^2 (C_1 t - \phi),$

$$= \tfrac{1}{2}(k'E_1E_o \frac{\Delta t}{2T_r})^2 \; [1 - \cos 2(C_1 t - \phi)] .$$

Thus a d-c output of $\frac{1}{2}(k'E_1E_o \frac{\Delta t}{2T_r})^2$ is realized.

II For C_1 negative (object receding)

Channel I (using $\cos(-\theta) = \cos\theta$)

$$E_{v1} = k'E_1E_o\left\{\frac{\Delta t}{2T_r}\cos(C_1\ t + \phi) + \frac{1}{2\pi}\sum_{n=1}^{N}\frac{\sin n\pi\frac{\Delta t}{T_r}}{n}\left(\cos\left[(\frac{2n\pi}{T_r} - C_1)t - \phi\right]\right.\right.$$

$$\left.\left. + \cos\left[(\frac{2n\pi}{T_r} + C_1)t + \phi\right]\right)\right\}.$$

Channel II (using $\sin(-\theta) = -\sin\theta$)

$$E_{v2} = k'E_1E_o\left\{\frac{\Delta t}{2T_r}\sin(C_1\ t + \phi) - \frac{1}{2\pi}\sum_{n=1}^{N}\frac{\sin n\pi\frac{\Delta t}{T_r}}{n}\left(\sin\left[(\frac{2n\pi}{T_r} - C_1)t - \phi\right]\right.\right.$$

$$\left.\left. - \sin\left[(\frac{2n\pi}{T_r} + C_1)t + \phi\right]\right)\right\}.$$

Advancing E_{v1} 90° in phase,

$$E'_{v1} = k'E_1E_o\left\{-\frac{\Delta t}{2T_r}\sin(C_1\ t + \phi) - \frac{1}{2\pi}\sum_{n=1}^{N}\frac{\sin n\pi\frac{\Delta t}{T_r}}{n}\left(\sin\left[(\frac{2n\pi}{T_r} - C_1)\ t - \phi\right]\right.\right.$$

$$\left.\left. + \sin\left[(\frac{2n\pi}{T_r} + C_1)t - \phi\right]\right)\right\}.$$

Now, by Method 1 (simple addition),

$$E_{v2} + E'_{v1} = -\frac{k_1'E_1E_o}{\pi}\sum_{n=1}^{N}\frac{\sin n\pi\frac{\Delta t}{T_r}}{n}\sin\left[(\frac{2n}{T_r} - C_1)t - \phi\right],$$

or, by Method 2,

$$E_{v2}E'_{v1} = -\left(k'_1E_1E_o \frac{\Delta t}{2T_r}\right)^2 \sin^2(C_1\, t + \phi),$$

$$= -\tfrac{1}{2}\left(k'_1E_1E_o \frac{\Delta t}{2T_r}\right)^2 \left[1 - \cos 2(C_1\, t - \phi)\right] \quad ,$$

and now a d-c output of $-\frac{1}{2}\left(k'_1E_1E_o \frac{\Delta t}{2T_r}\right)^2$ is obtained.

Thus, by Method 1, the upper sidebands of each of the original pulse frequencies are obtained with C_1 positive and the lower ones with C_1 negative. A pair or series of filters could then be used to indicate the sign of C_1 .

By Method 2, a d-c output is obtained whose polarity depends on the sign of C_1 . Since d-c is used as the output, this method will result in a very narrow effective bandwidth, with an attendant improvement in signal/noise ratio.

APPENDIX B

RESPONSE OF 3RD DETECTOR FOR A SINGLE PULSE

$$i_1\ (t) = 0 \qquad t < t_1 \quad \text{and} \quad t_1 + \Delta t < t < T,$$

$$i_1\ (t) = I_o \qquad t_1 < t < t_1 + \Delta t.$$

(See Fig. 7)

By a Laplace transform:

$$i_1\ (p) = I_o \frac{\varepsilon^{-t_1 p} - \varepsilon^{-(t_1 + \Delta t)p}}{p} .$$

The network transfer function is given by

$$e_2 = \frac{R_1}{1 + pC(R_1 + R_2)} i_1 .$$

Then

$$e_2(p) = R_1 I_o \frac{\varepsilon^{-t_1 p} - \varepsilon^{-(t_1 + \Delta t)p}}{p[1 + pC(R_1 + R)]} ,$$

and $e_2(t)$ can be given by the contour integral

$$e_2(t) = \frac{I_o R_1}{2\pi j} \int \frac{\left[\varepsilon^{-t_1 z} - \varepsilon^{-(t_1 + \Delta t)z}\right]\varepsilon^{zt}}{zC(R_1 + R)\left[z + \frac{1}{C(R_1 + R)}\right]} dz ,$$

where the contour encloses all poles.

Thus

$$e_2(t) = I_o R_1 \left[\varepsilon^{\frac{t_1 + \Delta t}{C(R_1 + R)}} - \varepsilon^{\frac{t_1}{C(R_1 + R)}}\right] \varepsilon^{-\frac{t}{C(R_1 + R)}} \quad \text{for } t > t_1 + \Delta t,$$

$$= I_o R_1 \left[1 - \varepsilon^{-\frac{t - t_1}{C(R_1 + R)}}\right] \quad \text{for } t_1 < t < t_1 + \Delta t,$$

$$= 0 \quad \text{for } t < t_1 .$$

Then at end of gate (time T)

$$e_2(T) = I_oR_1 \left[\varepsilon^{-\frac{T - (T_1 + \Delta T)}{C(R_1 + R)}} - \varepsilon^{-\frac{T - t_1}{C(R_1 + R)}} \right] \quad \text{for } T > t_1 + \Delta t,$$

$$= I_oR_1 \left[1 - \varepsilon^{-\frac{T - t_1}{C(R_1 + R)}} \right] \quad \text{for } t_1 < T < t_1 + \Delta t.$$

Let $\frac{T}{C(R_1 + R)} = k_1.$

$$\text{Then } \frac{e_2(T)}{I_oR_1} = \varepsilon^{-k_1(1 - \frac{t_1}{T})} \left[\varepsilon^{\frac{k_1 \Delta t}{T}} - 1 \right] \quad T > t_1 + \Delta t,$$

$$= \varepsilon^{-k_1} \left[\varepsilon^{k_1 \frac{\Delta t}{T}} - 1 \right] \quad t_1 < T < t_1 + \Delta t. \quad (2.8)$$