Eigen frequency statistics and excitation statistics in rooms: model tests with electrical waves

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Publisher's version / Version de l'éditeur:
https://doi.org/10.4224/20331430
Technical Translation (National Research Council of Canada); no. NRC-TT-1261, 1966

PREFACE

The mathematics underlying the distribution of natural modes of acoustical vibrations in a room is the same as that for electromagnetic eigen functions in a resonant enclosure.

This paper concentrates almost entirely on the electrical case and discusses both the relative amplitudes and the frequency spacing between eigen functions. The theoretical deductions are supported by experimental observations on these quantities. It is shown that when the ratio of enclosure length to wavelength is around 6 : 1 the degree of degeneracy of modes in a perfect cube is not as great as one would expect from theoretical considerations.

This is taken to mean that the irregularities in the construction of the cavity introduce a degree of randomness. When the cube is deformed by about 1.0% so that the sides originally 200 mm x 200 mm x 200 mm become 200 mm x 199 mm x 198 mm, the randomness becomes complete. Even the inherent degeneracy of 2 due to the two possible directions of polarisation is removed due to a further degree of randomness being impressed on the assumed Poisson type distribution of the mode separation.

The other measure of randomness is provided by the relative amplitudes of the different eigen functions. A deliberate amount of degeneracy is introduced by actuating the modes in the centre of one face. By introducing irregularities of the order of the cube of the wavelength, the degree of randomness approaches its theoretical maximum.

All the conclusions reached for the electromagnetic eigen functions should hold for the acoustical case. The only modification which must be made to all the deductions is the removal of the polarisation degeneracy.

The Division of Building Research wishes to record its thanks to Mr. D.A. Sinclair, Head, Translations Section, National Research Council, for translating this paper and to Dr. R.J. Donato who checked the translation.

Ottawa
October 1966

R.F. Legget
Director

NATIONAL RESEARCH COUNCIL OF CANADA

TECHNICAL TRANSLATION 1261

Title: Eigen frequency statistics and excitation statistics in rooms. Model tests with electrical waves

(Eigenfrequenzstatistik und Anregungsstatistik in Räumen. Modellversuche mit elektrischen Wellen)

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Reference: Acustica, 4: 457 - 468, 1954

Translator: D.A. Sinclair, Translations Section, National Science Library

EIGEN FREQUENCY STATISTICS AND EXCITATION STATISTICS IN ROOMS

MODEL TESTS WITH ELECTRICAL WAVES

Abstract

In the first part of the present paper the natural frequency spacing statistics of resonant cavities is investigated. It is shown that the non-accidental degenerations of regular rooms can be effectively split up by minute deviations from geometric symmetry. With a cube of 20 cm length, for instance, the average degree of degeneration is 24 at 3 cm wavelength. However, a difference between the three dimensions of 0.1 cm only suffices to make the spacing statistics a purely random one.

In the second part the excitation statistics of the normal modes is investigated. Here again a degenerate case is the starting point. A rectangular cavity is excited at the central point of one of its surfaces. In this manner only one fourth of the normal modes will be excited. But introducing a tiny perturbing element with a volume of $(\text{wavelength})^3$ into the resonator results in all the modes being excited according to a purely random law.

Though the present investigation is of interest within the domain of room acoustics predominantly, all measurements have been performed with microwaves in metallic cavities, because of their high Q which is in the order of some 10^4. Thus the individual resonances can be resolved even at higher ratios of room dimensions to wavelength.

1. Introduction

The frequency curve of a room can be regarded as a superposition of the individual resonance curves. It follows that the frequency curve is determined by three distributions, namely the statistics of the spacing, the excitation and the damping of the individual eigen frequencies. In the present study the spacing and excitation statistics are investigated.

It is shown that for all actual comparatively large rooms the spacing and excitation statistics are those of a completely random space. Accordingly, the variations of the frequency curve of a room (mean height of the "peaks", mean distance between maxima, etc.) become, in the limiting case of "half-value width large compared with the mean distance between eigen frequencies" (EF) and "linear dimensions of the room large compared with the wavelength", a function of the reverberation time only.

There is no difficulty in transferring the results obtained from electrical resonant cavities to corresponding acoustical rooms, provided it is kept in mind that the number of EF in a given interval is about twice as great as in the analogous acoustical case. This is because of the transversal character of electromagnetic waves, for the complete description of which it is necessary to specify the direction of polarization as well as the data needed for the description of a sound wave in air. For a rectangular space the EF densities can be given as a function of the volume V, the area S and the edge length L:

Number of acoustical EF $= \left(\frac{4\pi V}{\lambda^3} + \frac{\pi S}{2\lambda^2} + \frac{L}{8\lambda}\right)\cdot\frac{\Delta f}{f}$; (1a)

number of electrical EF $= \left(\frac{8\pi V}{\lambda^3} - \frac{L}{4\lambda}\right)\cdot\frac{\Delta f}{f}$, (1b)

where λ is the wavelength and $\Delta f/f$ the relative size of the frequency interval.

For the number of electrical EF we here consider only the term given by the volume. The correction resulting from addition of the edge length term is only about -1%. Formula (1b) can then be written in the following simple form

$$\frac{\overline{df}}{f} = \frac{\lambda^3}{8\pi V}, \tag{2}$$

where $\overline{df}/f$ is the mean relative interval between neighbouring EF.

Another difference from acoustics consists in the fact that in the electrical case both field values are vectors. For sound in air there is a scalar field value, the sound pressure. As a consequence, for a rectangular acoustic resonator there is a possibility of exciting all EF uniformly, since all natural vibrations show a bulging of the sound pressure in the corners. With an electrical resonant cavity, however, it is impossible to produce an excitation at one point such that all EF are uniformly excited.

2. The Spacing Statistics of the Eigen Frequencies

(a) Definition of a spacing parameter

As a characteristic value for the more or less uniform distribution of the EF the second moment of the spacing statistics between each pair of neighbouring EF has been introduced by R.H. Bolt[1,2], and is referred to the mean interval determined principally by the volume. He denotes this value by Ψ:

$$\Psi = \frac{\frac{1}{N}\sum df^2}{\left(\frac{1}{N}\sum df\right)^2} = \frac{N}{\Delta f^2}\cdot\sum df^2,$$

where df is the distance between neighbouring EF, Δf the size of the interval in question and N the number of EF in this interval.

When the intervals between the EF are equal, then according to this definition $\Psi = 1$. for a completely random distribution, where the occurrence of an interval df has a probability proportional to $\exp(-df/\overline{df})$, $\Psi = 2$. In rooms with a high degree of geometric symmetry (spheres, cubes, ellipsoids of rotation, bodies with square cross-sections, cylinders, etc.) non-accident degenerations occur. A certain number of EF coincide, and leave correspondingly large gaps on the frequency axis. The spacing index of the EF, Ψ, is large, since in addition to the frequency interval 0, fairly large intervals occur more frequently than would be the case for a pure random distribution. The question is, how far must the shape of the resonator depart from geometrical symmetry in order to render the distribution of the EF random. Quantitatively speaking, the degree of randomness is measured by the closeness of the approach of Ψ to the value 2.

It should be mentioned that it is not always possible to infer the existence of pure random distribution from the condition $\Psi = 2$. In a cylindrical resonator, for example, the cross-section of which can be converted into itself by a rotation with an angle of less than 180°, and which is operated in the fundamental vibration form, the EF are degenerate and are equidistant. In the frequency range in which the resonator can vibrate only in the fundamental vibration form, this also gives $\Psi = 2$, even though there can be no question here of randomness of the EF distribution.

(b) Results

The measurements were begun with a rectangular cavity of dimensions 17 x 27 x 43 cm³, corresponding to an edge length ratio of approximately 2 : 3 : 5. 127 EF were counted in an interval of 100 MHz at $\lambda = 3.2$ cm. The theoretical number of EF for this interval is N = 157. (Thus 30 EF were not found). The sum of the squares of the intervals was $\sum df^2 = 126$ MHz. The spacing index thus becomes

$$\Psi = \frac{N}{\Delta f^2} \sum df^2 = 1.98 .$$

This result implies, of course, that all EF not found coincide exactly with other EF, so that the intervals still to be added are 0 intervals. This is certainly only approximately true. The true value of Ψ for the evaluated interval is thus somewhat lower.

The result $\Psi \approx 2$ shows that the distribution of the EF in the rectangular space in question is random, for $\Psi = 2$ is the value for a totally random distribution. What is surprising about this result is the fact that the rectangular space is nevertheless still a very regular body. The original explanation for this phenomenon was the fact that the dimensions of the rectangle had been chosen in accordance with the ratio $1 : \sqrt[3]{4} : \sqrt[3]{16}$, or approximately 2 : 3 : 5 (3).

In order to clarify the situation, measurements were then carried out in a cube, and led to no less astonishing results. As a consequence the above test result must be interpreted as follows: the slight departures of the side ratios from the simple whole numbers 2 : 3 : 5, combined with the slight geometrical irregularities of the resonator are by themselves sufficient to produce a completely random distribution of the EF. The theory of the magical ratio 2 : 3 : 5 has no significances here in view of the given ratio of wavelength to edge length of 1 : 6. In the case of rectangular spaces the value of Ψ is determined rather by the extent to which the side ratios depart from small whole numbers. This becomes even clearer when we consider the test results on the cube.

The most striking phenomenon in the EF distribution of a cube is the fact that there is a minimum interval between any two non-coincident EF. The EF of a cube are given by the formula

$$f = \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2}\,,$$

where c is the velocity of light, a the side length of the cube and n_x, n_y, n_z the three wave indices, giving the number of half periods in the direction of the three edges.

The sum of the squares within the root is a whole number. The minimum distance Df between two non-coinciding EF is given by the increase of this sum by unity. Hence, for the relative minimum interval we immediately get

$$\frac{Df}{f} = \frac{1}{2}\cdot\frac{1}{n_x^2+n_y^2+n_z^2} = \frac{1}{8}\left(\frac{\lambda}{a}\right)^2. \tag{3a}$$

Comparing equation (2) for the mean interval of EF with this,

$$\frac{\overline{d}f}{f} = \frac{\lambda^3}{8\pi V}\,,$$

it will be recognized that on the average at least

$$\frac{Df}{\overline{d}f} = \pi\frac{a}{\lambda}\,. \tag{3b}$$

EF must coincide. In fact, the mean degree of degeneration is somewhat greater, since even intervals 2 · Df occur. This is always the case whenever a whole number cannot be represented as the sum of three squares. As the theory of numbers shows[4], that is the case in the limit for every sixth whole number. When the numbers are large thus the mean degree of degeneration is greater than the value of $Df/\overline{d}f$ by a factor of 6/5:

Mean degree of degeneration of a cube $= \frac{6\pi}{5}\cdot\frac{a}{\lambda}$. (3c)

The spacing index is then obtained directly from the definition

$$\Psi_{\text{CUBE}} = \frac{4\pi}{3}\cdot\frac{a}{\lambda}. \tag{3d}$$

In Table I the possible values of n^2 and degrees of degeneration are given for the interval

$$n^2 = n_x^2 + n_y^2 + n_z^2 = 150 \quad \text{TO} \quad n^2 = 162.$$

It will be noted that in general the degree of degeneration is twice as great in the electrical case as in the acoustical, since there are two possible polarizations for the oblique electrical waves, for which none of the three wave indices n_x, n_y, n_z vanishes.

Under the column "permutations" is given the number of possible permutations of the three indices. Generally speaking this is six, if two indices are the same, however, it is reduced to three.

According to Table I the total number of electrical EF for this interval is 249. According to formula (1b), 254 EF should fall within this interval, and according to the simplified formula (2), 255. The total number of acoustical EF for the same interval according to Table I is 135. Formula (1a) gives the value 143.

Of the 13 indices tabulated, 10 are occupied by EF. The average degeneration, therefore, is 24.9 in the electrical case. According to formula (3c) the mean degree of degeneration for the region about $n^2 = 156$ should be 23.5. It is further noted that the highest degree of degeneration, 48, is just twice the mean. R.H. Bolt(1) has already pointed this out.

The EF groups listed in the table were measured in a cube of 20 cm side in a region about the wavelength 3.2 cm. The results are assembled in Table II. The experimental apparatus will be described below. The second column again gives the degree of degeneration as obtained from Table I. In the next column we enter the number of EF actually found, which, of course, is still considerably smaller, since the EF are decidedly grouped. Nevertheless more EF were found in more densely occupied places than for more sparsely occupied ones. The 6 EF belonging to the index $n^2 = 160$ were actually all detected. In the next column the spread of the individual groups is recorded, as caused by the slight mechanical tolerances in the manufacture of the cube. Finally, the last column gives the interval between the adjacent EF of two neighbouring groups.

The measured groups cover a wavelength region of 3.268 - 3.143 cm, corresponding to a frequency interval of 360 MHz. By summation of the spreads and the intervals between groups from $n^2 = 150$ to 161 in accordance with the last two columns of Table II, we obtain 354 MHz. This provides a check on the accuracy with which the individual frequency intervals have been measured.

If we assume that the EF are situated randomly within the individual groups, then from the above test results the spacing index of the intervals is obtained as $\Psi = 16$. We are led to assume randomness within the individual groups by observing the spectra on the screen of the cathode ray tube. This assumption, however, is not essential and has no great influence on the value of Ψ, which is determined chiefly by the great distances between the groups.

The theoretical spacing index of the EF is calculated for the present case from Table I and gives $\Psi = 27.8$. According to formula (3d) it should be 26.2.

At first glance the difference between the measured $\Psi = 16$ and the theoretical value of 27 is disconcerting. Here we already see how very sensitively the spacing index reacts to small departures from the ideal geometrical shape. The mechanical tolerances are of the order of a few tenths of a millimetre. With a side of 20 cm, this comes to only about 0.1%. The influ- of the excitation on the other hand is negligible, as can be established by comparisons for excitation of different intensity.

The oscillograms of Figures 1 and 2 are a good illustration of these conditions. Fig. 1a shows the 24 EF of the group $n^2 = 158$, of which, however, only 12 are really recognizable. Figures 1b to 1d show the same group for different contractions of two opposite faces of the cube. In Fig. 1d to the right the transition to the group $n^2 = 157$ is already made (higher frequencies are plotted to the left), although the "effective contraction" is only about 0.25 mm. (The effective contraction is defined by the change of volume divided by the area of the squeezed face.)

In order to close the double gaps as well, i.e. in order to get a transition here to the group $n^2 = 160$, contraction would have to be twice as great. If the group were not already divided up, the necessary change would of course be somewhat greater still. For the closing of the double gaps $2 \cdot Df$ of a mathematical cube a relative change of side of $2 \cdot Df/f = (\lambda/2a)^2$ is required.

On contraction, the EF form into ever new combinations, so that the oscillogram changes continuously. Only a few isolated EF remain intact, i.e. can be differentiated, like the one furthest to the left in Fig. 1. This one is recognizable in all four oscillograms.

Figure 2 shows four oscillograms of the 6 EF's of the group $n^2 = 160$. In this rare case all the EF are visible and can be differentiated. Figures 2b to 2d show the group for a slight contraction in each of the three edge directions. It will be seen that two of the 6 EF are always displaced towards higher frequencies (to the left), while the other four are scarcely affected. This is because the 6 EF of the group $n^2 = 160$ are not merely tangential waves, because one of the three waves indices (12, 4, 0) vanishes, but are also quasi-axial, since one of the two non-vanishing indices (4) is considerably smaller than the other (12). The angles between the wave vector and the three edge directions are 18°, 72° and 90°.

Following the measurement of the cube, the EF statistics of a near cube with the dimensions 200 x 200 x 199 mm^3 were recorded. The change of side length by 1 mm, corresponding to 0.5%, should, on the basis of the above considerations, be enough to cause the widest gaps to vanish. In an interval of 355 MHz, 151 EF were found. The theoretical number is 231. The sum of the squares of the intervals was 1966 MHz^2. Assuming that the EF not found coincide exactly with others, then the missing intervals are null intervals, so that as an upper boundary for the relative spacing index we have $\Psi_u = 3.6$. However, if it be assumed that the EF not found do not coincide with the others, then the value of Ψ is reduced by the ratio 151/231. This yields, as a lower limit for the spacing index, $\Psi_l = 2.4$ (in this connection see Section 2 (c)).

The truth lies between these extremes. True, the EF not found do not coincide exactly with the others, but they will preferably lie close to them. The further an EF is situated from the neighbouring one, the less likely it is that it will be accidentally concealed by a more strongly excited one. The greater, therefore, is the probability that it will be found. If we assume that the EF not found are situated at a mean distance of 0.8 MHz from their nearest neighbours, then the most probable value for the index of fluctuation is given by:

$$\Psi = \Psi_u - 2\frac{N - N'}{N'} \cdot \frac{\overline{\mathrm{d}s}}{\overline{\mathrm{d}f}} = 3.1,$$

where N is the theoretical number of EF, N' the number of EF found, $\overline{ds}$ the mean distance between the EF not found and their nearest neighbours, $\overline{df}$ the theoretical mean distance between EF. The formula will be derived in Section 2 (c).

It will be seen that Ψ is reduced to almost one tenth of the theoretical value by the reduction of a side from 200 mm to 199 mm. For this, of course, the mechanical tolerances, i.e. principally the irregularities in the boundary faces of the cube, are in part responsible. For in general four EF should still coincide, and even for otherwise random distribution this should yield a value of $\Psi = 4 \cdot 2 = 8$. (Since two sides are equal, there are two possible permutations. For the oblique waves present in the remainder a further factor of 2 is added on account of the two independent polarizations.) The mechanical tolerances also offset the still remaining degenerations. However, a definite departure from the value of the purely accidental distribution $\Psi = 2$ can still be discerned.

Two other near cubes with dimensions 200 x 200 x 198 mm^3 and 200 x 199 x 198 mm^3 were then evaluated. The first one is similar to the one just discussed except that one side is still further reduced from 199 to 198 mm. The second near cube now retains only the symmetry of a general parallelepiped.

Measurements on the near cube of 200 x 200 x 198 mm^3 in an interval of 97 MHz reveal 44 EF. The theoretical number is 63. The sum of the squares of the intervals was 543 MHz^2. Thus as an upper boundary we get $\Psi_u = 3.6$ and the lower boundary $\Psi_l = 2.5$. The most probable value obtained with the formula employed above is $\Psi = 3.2$. As expected, the statistics of the near cube 200 x 200 x 198 mm^3 do not differ from that with the dimensions 200 x 200 x 199 mm^3. Indeed the agreement is even closer than would be expected in view of the statistical fluctuations of finite test intervals.

In the parallelepiped 200 x 199 x 198 mm^3 an interval of 671 MHz was measured. 230 of the theoretical 420 EF were found. The sum of the squares of the intervals was 3055 MHz^2. As boundaries for the index of fluctuation, therefore we get $\Psi_u = 2.8$ and $\Psi_l = 1.6$.

The assumption that the true value of $\Psi = 2$, is confirmed by comparing the relative frequencies of the individual intervals with the theoretical frequency distribution of intervals for random positions of the EF. This is done in Fig. 3. The smooth curve is the theoretical distribution, which runs proportionally to $\exp(-df/\overline{df})$ with $\overline{df} = 1.6$ MHz. The broken line gives the measured frequencies taken in groups of 0.5 MHz width. A good agreement is noted above 2 MHz, whereas for smaller frequency intervals the number of measured intervals lags farther and farther behind the theoretical number. This is the "concealing effect" already mentioned: weakly excited EF are no longer detectable in the immediate vicinity of more strongly excited ones.

The result $\Psi = 2$ for the parallelepiped with dimensions 200 x 199 x 198 mm^3 still requires some explanation. For an otherwise random distribution of the EF the value of Ψ should actually be about four, owing to the degeneration of polarization always present in the parallelepiped. It is easily realized, however, that the slightest irregularities of the boundary faces again eliminate this cause of degeneration. For the mathematical parallelepiped two EF always coincide in the oblique waves. If there is no other symmetry (the sides show no simple relationship with each other), the spacing statistics of these simply degenerated EF obey the law of chance, with a mean interval, of course, of $2 \cdot \overline{df}$. In order to eliminate this regularity, the EF need merely be displaced relative to each other randomly up to $2 \cdot \overline{df}$. This can be brought about by mechanical tolerances of the order of $a \cdot \overline{df}/f$.

In the present instance these are approximately 0.03 mm. Since the irregularities of the last measured parallelepiped are almost an order of magnitude greater, we can certainly conclude that the spacing index is equal to two in this case. This finding will be applied below to an estimation of the masking width $\overline{ds}$.

From the above measurements on a parallelepiped, a cube, and three near cubes, it can be concluded that the distribution of EF in every actually occurring space of dimensions greater than a few wavelengths follows the law of chance. Moreover, since the phenomenon of polarization degeneration does not occur in the case of air-transmitted sound, the boundary of the space in the acoustical case can even consist of mathematically smooth surfaces.

(c) An estimate of the fluctuation square

According to the definition given above of the relative fluctuation square Ψ of the spacing statistics of the EF, we may write

$$\Psi = \frac{N}{\Delta f^2} \sum^{N} \mathrm{d}f^2,$$

where $\Delta f = \sum \mathrm{d}f$ is the size of the measured interval. The index "N" over the summation sign indicates that the sum in this formula has "N" terms.

In reality, however, only $N' < N$ intervals were measured and some of them wrongly, that is to say when they were divided up by EF which were not found. In place of the sum $\sum^{N} \mathrm{d}f^2$, only the expression $\sum^{N'} \mathrm{d}f'^2$ is known, where df' represents the measured intervals which agree only partially with the true intervals df.

Now if the intervals not found are equal to zero, then

$$\sum^{N} \mathrm{d}f^2 = \sum^{N'} \mathrm{d}f'^2.$$

This assumption is tantamount to saying that the EF not found coincide exactly with other EF, and leads to an upper limit for the fluctuation square:

$$\Psi_u = \frac{N}{\Delta f^2} \sum^{N'} \mathrm{d}f'^2. \tag{4a}$$

A lower boundary for the fluctuation square is obtained from the assumption that the EF not found are independent of the others and the spacing statistics remain the same owing to their being added. Then

$$\sum^{N} \mathrm{d}f^2 = \frac{N}{N'} \sum^{N'} \left(\frac{N'}{N} \mathrm{d}f'\right)^2 = \frac{N'}{N} \sum^{N'} \mathrm{d}f'^2$$

and consequently,

$$\Psi_l = \frac{N'}{\Delta f^2} \sum^{N'} df'^2. \tag{4b}$$

In reality the EF not found are neither entirely independent of those that are found, nor do they coincide exactly with these. Rather they are situated with a certain distribution in the vicinity of more strongly excited ones. This is shown very strikingly by Fig. 3. The intervals df > 2MHz have all been found. Of the other intervals, the smaller they are the more of them are missing.

We are thus able to calculate Ψ more accurately than from the above boundary values Ψ_u and Ψ_l. If we denote the distances from the EF not found to the nearest neighbours by ds, then we may write:

$$\sum^{N} df^2 = \sum^{2N'-N} df'^2 + \sum^{N-N'} (df' - ds)^2 + \sum^{N-N'} ds^2.$$

The three terms on the right side state that of the N' measured intervals N' - (N - N') = 2N' - N remain intact. N - N' intervals are reduced by ds and N - N' intervals of the size ds are newly added.

This theorem, of course, implies that no many of the newly added EF fall in the same gaps. The smaller N - N' is compared with N, the better satisfied this condition is. Assuming further that the new intervals are small compared with measured df', then $\sum ds^2$ can be neglected compared with $\sum df' \cdot ds$ and we obtain

$$\sum^{N} df^2 = \sum^{N'} df'^2 - 2 \sum^{N-N'} df' \cdot ds,$$

or, if the distributions of df' and ds are uncorrelated:

$$\sum^{N} df^2 = \sum^{N'} df'^2 - 2(N - N') \cdot \overline{df'} \cdot \overline{ds}.$$

From these sums we again obtain the relative fluctuation squares by normalizing:

$$\Psi = \Psi_u - 2 \frac{(N - N')}{N'} \cdot \frac{\overline{ds}}{\overline{df}}.$$

In the case of the parallelepiped measured above, this formula can be used for the calculation of the masking width $\overline{ds}$. With $\Psi = 2$, $\Psi_u = 2.8$, N = 420, N' = 230 and $\overline{df}$ = 1.6 MHz, then

$$\overline{ds} = 0.8\,\text{MHz}.$$

This value for the masking width, the mean value of the EF not found, would also be assumed by consideration of Fig. 3. $\overline{ds} = 0.8$ MHz also fits well for the average half-value width of the EF, which was determined with 0.5 MHz. The average half-value width of the more strongly excited EF, which of course are the principal causes of the masking effect, is somewhat greater still. The greatest half-value widths are situated even at 1.4 MHz.

In the corrections of the Ψ values in the preceding section this value for the masking width $\overline{ds} = 0.8$ MHz was always employed. The transfer of this value of $\overline{ds}$ to other resonators is valid if the average half-value width of the EF is the same and the volumes (EF densities) are not too different.

3. The Excitation Statistics of the Natural Vibrations

(a) Definition of an excitation parameter

As an additional parameter of the wave theory, on which the frequency curve of a space is based, the excitation statistics of the natural vibrations have been investigated. Similar to Ψ above, we now define a relative fluctuation square of the excitation;

$$\Phi = \frac{\frac{1}{N}\sum a^2}{\left(\frac{1}{N}\sum a\right)^2},$$

where a is a measure of the excitation intensity of the individual natural vibrations, and specifically $a^2 \sim P_{ab}$, of the energy absorbed for the EF in question.

For example, if a rectangular cavity resonator is excited at the centre of one side, only a quarter of all the EF are involved. From the definition given above it follows immediately that Φ is comparatively large, and in any case greater than 4. $\Phi = 4$ only for the case that the EF involved are all equally strong.

This case, which is degenerate with respect to the excitation, can be eliminated by interferences which destroy the original symmetry of the arrangement. The new natural vibrations are then combinations of all natural vibration forms, including those not previously excited. The number of resonances in a given frequency interval is increased by a factor of 4.

If one of these EF is excited, then no longer will only one or two vibration forms of the parallelepiped be excited with a uniform wave vector, as is the case in the undisturbed rectangular box, but an ever greater number simultaneously, the wave vectors of which are pointed in various directions. This wave theory phenomenon of the loss of certain distinctive directions is not to be confused with the concept of diffuseness or direction diffuse-[3,5-9] ness used elsewhere in the acoustics of space.

(b) Results

The measurements were carried out on an electromagnetic cavity resonator with dimensions 101 x 150 x 299 mm^3. For the fluctuation square of the intervals of the EF the value $\Psi = 2.2$ is obtained. For the round numbers 100 x 150 x 300 mm^3, approximately $\Psi = 10$ would have been obtained.

First two measurements were made of the excitation statistics in the undisturbed space. The interval employed for all measurements was chosen such that the theoretical number of EF was 200. Actually 56 and 54 EF were observed respectively, i.e. somewhat more than expected. This was due, of course, to unavoidable departures from perfect symmetry. The excitation parameter in one case was $\Phi = 4.96$, and in the other $\Phi = 5.04$. The two measurements were made with coupling intensities between resonator and generator differing greatly from each other. The small influence on the result of the measurement will be recognized.

In order to have a better comparison with the theory, the 6 and 4 weakest EF, respectively, can be disregarded. We then get $\Phi = 5.22$ and 5.18, respectively. The theoretical value 5.33, which will be derived below, is of course unobtainable, because the neglect of the weakest EF is not perfect compensation for the given departure from mathematical symmetry. The agreement with the theory is nevertheless good.

The question now arises as to how the interferences of the surface of the resonator must be made in order to mix substantially the EF of the parallelepiped, thereby improving the excitation statistics. In this connection the theory, which is outlined further below, states:

1. The critical interference volume δV is of the order of λ^3, regardless of the size of the space.
2. The effect of the interference with respect to the mixing of the EF is independent of the geometrical form of the interference, provided its length is small compared with the wavelength.
3. The local position and distribution of the interference likewise has no effect on the mixing, unless the original symmetry of the set-up is only partially destroyed by the interference.

In what follows we shall first mention a series of experiments that confirm these three theoretical results. In Fig. 4 the test results for Φ are plotted against the interference volume δV. The open circles denote measurements with small cubes which were placed on the floor of the resonator as interfering bodies. The filled-in circles give the measurements with circular cones. The small square denotes a measuring point at which the cube is divided into eight equal parts and distributed over the space. The two circles on the ordinate of the diagram are the two measurements already cited for the empty box. The two boundary tangents $\Phi = 5.33$ for the empty box and $\Phi = 1.57$ for an excitation statistic of pure chance are also included in the drawing. As is evident from Fig. 4, is a function of the interference volume δV only. For $\delta V > \lambda^3/2$ the value of Φ is stationary. The departure from the theoretical value 1.57 is only 3%.

In particular the independence of of the position of the interference has been confirmed, which in each case has been chosen purely accidentally by arbitrary placing the interfering element into the space. Moreover, three more measurements with the λ^3 cube have been made in this connection. The results were $\Phi = 1.60$, 1.62 and 1.64, respectively. The remaining fluctuations are of a statistical nature. They would vanish only for the measurement of an infinitely large interval.

However, if the position of the interference element is deliberately preselected in such a way that the original symmetry of the space remains wholly or partially intact, then only an imperfect intermingling of the EF occurs. Φ is greater than for an equal interference volume but asymmetrical position. To illustrate this a measurement with the λ^3 cube has been made, where the cube was placed in the centre of a plane perpendicular to the excitation face. As a consequence only one of the two mirror symmetries in the plane of excitation is destroyed. Instead of about 50 EF which would be found in the empty space, in this case about twice that number (exactly 103) EF were obtained. For the fluctuation parameter of the excitation we got $\Phi = 2.77$. The theoretical value for satisfactory symmetry is $\Phi = 2 \cdot 1.57 = \Phi$ 3.14.

(c) Theory of the excitation parameter

According to the definition given above the excitation intensity a is measured by the energy P_{ab} absorbed for the EF involved. For the energy absorbed by the resonator, however, we have(13):

$$P_{ab} = P_{in\,out} \frac{2d_r}{d_r + d_0}.$$

Here d_o is the internal damping of the resonator and d_r the damping due to the coupling. P_{in} is the energy received from the generator by radiation. If P_{in} is kept constant and the coupling is weak ($d_r \ll d_o$), both of which conditions are easily realized, the fluctuation of the internal damping of the different EF is relatively small, and the simple relationship $a^2 \sim d_r$ holds. (The fluctuation of d_o was measured on 108 EF. The result was $\overline{d_o^2} - \overline{d_o}^2 = 0.07 \cdot \overline{d_o}^2$. The correction of Φ when the fluctuation of d_o was taken into account was approximately 3%).

If the coupling takes place through a hole at the end of a hollow conductor which conducts only the fundamental wave, this can be regarded as a magnetic dipole. The coupling strength is proportional to the component H_y of the magnetic field of the natural vibration involved parallel to the exciting field in the hole[14]. For the coupling damping, therefore, we may write $d_r \sim H_y^2$. The relative fluctuation square of the excitation thus becomes:

$$\Phi = \frac{\overline{a^2}}{\overline{a}^2} = \frac{\overline{H_y^2}}{\overline{|H_y|}^2}.$$

In case of an undistorted parallelepiped which is excited at the centre of a side, we may now write:

$$H_y = \begin{cases} 0 & \text{for } \tfrac{3}{4} \text{ of all EF,} \\ \sin\vartheta \cdot \cos\varphi & \text{for the remaining } \tfrac{1}{4}. \end{cases}$$

Here ϑ is the angle between the wave vector of the natural vibration involved and the y direction, the direction of the exciting magnetic field, and φ the angle of polarization.

In forming the average, ϑ should be averaged over all space directions and φ over all polarizations:

$$\overline{H_y^2} = \frac{2}{3} \cdot \frac{1}{2}\ \frac{1}{4} = \frac{1}{12},$$

$$\overline{|H_y|} = \frac{\pi}{4} \cdot \frac{2}{\pi}\ \frac{1}{4} = \frac{1}{8}.$$

Thus, for the undistorted parallelepiped

$$\Phi_0 = \frac{16}{3} = 5{,}33\ldots.$$

In the wholly random case the component H_y is made up of a large number of independent components. The H_y values are thus distributed according to Gauss:

$$W(H_y) \sim \exp(-H_y^2/2 \cdot \overline{H_y^2}).$$

Hence, we get directly

$$\overline{|H_y|}^2 = \frac{2}{\pi} \cdot \overline{H_y^2},$$

or, for the fluctuation parameter in the limiting case of strong mixing:

$$\Phi_\infty = \frac{\pi}{2} \doteq 1{,}57\ldots.$$

Finally, we shall take up the question of how the interference volume δV (the variation of the surface of the parallelepiped) must be produced in order to attain the state of randomness characterized by $\Phi = 1.57$.

According to a known result of the time-independent theory of distortion(12), the variation of the eigen functions φ_i of the boundary value problem

$$\Delta\varphi + k^2\varphi = 0, \quad \left(\frac{\partial\varphi}{\partial n}\right)_{\text{BOUNDARY}} = 0$$

are represented as follows:

$$\varphi_i = \varphi_i^0 + \sum_j{}' b_{ij} \cdot \varphi_j^0.$$

If the distortion consists in a deformation of the surface δr, then the coefficients are

$$b_{ij} = \frac{1}{k_i^{0^2} - k_j^{0^2}} \int\int \delta r \, \varphi_j^{0*} \frac{\partial^2}{\partial n^2} \varphi_i^0 \, dS,$$

assuming that the undistorted eigen functions are normalized:

$$\int\int\int |\varphi_i^0|^2 \, dV = 1.$$

The $k_i^{0^2}$ values are the eigen values belonging to the φ_i^0 values. The first integral should be extended over the area S, the second over the volume V of the resonator.

If the distortion is concentrated in a region whose volume is small compared with the wavelength, we may write, more simply,

$$b_{ij} = \frac{1}{k_i^{0^2} - k_j^{0^2}} \varphi_j^{0*} \frac{\partial^2}{\partial n^2} \varphi_i^0 \, \delta V.$$

Here the eigen functions are to be taken at the place of the distortion. The independence of b_{ij} from the form of the distortion will be recognized.

Now, forming the mean value of $|b_{ij}|^2$ for a constant difference of eigen values, then, if the place possesses no special properties of symmetry, by neglecting numerical factors of the order of unity and observing the standardization $\overline{|\varphi_i^0|^2} = 1/V$, we obtain

$$\overline{|b_{ij}|} = \frac{k_i^{0^2}}{k_j^{0^2} - k_i^{0^2}} \cdot \frac{\delta V}{V}.$$

It will be realized that for this mean value formation a specific position of the distortion has disappeared. The mixing coefficients are now only a function of the size of the distortion volume and the distance between the two eigen values involved.

Exactly the same result would be obtained if the distortions were not concentrated in one place, but were distributed over several positions; assuming that all partial distortions are again small in their geometric volume compared with the wavelength.

Going from the eigen values $k_i^{0^2}$ to the frequency scale and denoting the difference of EF with Δf_{ij}, the above formula acquires the following still simpler form

$$\overline{|b_{ij}|} = \frac{f}{2\Delta f_{ij}} \cdot \frac{\delta V}{V}. \tag{5a}$$

This formula permits an interesting interpretation if the distorted volume δV is replaced by the mean detuning caused by the distortion. For the variation of eigen values, that is, we may write

$$k_i^2 = k_i^{02} + \iint \delta r\, \varphi_i^{0*} \frac{\partial^2}{\partial n^2} \varphi_i^0\, dS,$$

or in the case of concentrated distortion:

$$k_i^2 = k_i^{02} + \varphi_i^{0*} \frac{\partial^2}{\partial n^2} \varphi_i^0 \delta V.$$

Forming the mean value gives

$$\overline{|k_i^2 - k_i^{02}|} = \frac{2}{3} k_i^{02} \frac{\delta V}{V}.$$

Finally going over to the frequency scale, we obtain the known formula

$$\frac{\overline{\delta f}}{f} = \frac{1}{3} \frac{\delta V}{V}.$$

Substituting this relation between distortion volume δV and mean detuning $\overline{\delta f}$ in equation (5a), and neglecting the numerical factor which is close to unity, we obtain

$$\overline{|b_{ij}|} = \frac{\overline{\delta f}}{\Delta f_{ij}}. \tag{5b}$$

In this form it is clear that precisely those EF whose distance apart is less than the mean detuning are the ones that are intermingled.

However, replacing the Δf_{ij} values in equation (5a) by the mean interval between two EF according to formula (2), and requiring $\overline{|b_{ij}|} = 1$, then as a critical value for the distorted volume for the intermingling of neighbouring natural vibrations, we get

$$\delta V = \frac{1}{4\pi} \lambda^3. \tag{5c}$$

Essentially then, the required distorted volume does not depend on the dimensions of the space but only on the wavelength.

This result holds even in the case of damped natural vibrations, even if the half-value width of the resonance curves is large compared with the mean interval between EF. The relative half-value, however, must be small compared with unity.

Summing up the results of the investigation of the statistics of intervals and excitation of EF once more, it may be said that for every relatively large occurring space both these statistics follow the law of chance. The necessary departures from mathemtical symmetry are implied here in the three words "occurring in practice". Assuming that the EF have uniform half-width, or that the half-width varies but little, it follows that the statistical parameters of the frequency curve of such a space (mean height of the "peak", mean distance maxima, etc.) are given by the reverberation time alone.

If the latter condition is not satisfied, it must be expected that, for example, the mean distance between maxima of the frequency curve depends also on the fluctuation of the reverberation time[10-11].

4. The Apparatus

Figure 5 shows a photograph of the microwave part of the apparatus for measuring the excitation statistics. The Klystron generator for wavelengths of about 3.2 cm will be recognized. The Klystron passes its energy through a (not shown) attenuator to a so-called "magic tee"[15-16]. Here the energy is divided into two parts on the two side arms. Fifty percent vanishes in the non-reflecting cut-off at the left of the picture. The other half passes to the resonator, which is coupled on by a diaphragm. Outside of the resonances all energy is reflected, while at resonance a considerable part of the energy is absorbed. The remaining energy returns to the magic tee, where another division takes place. Fifty percent goes into the generator arm and is absorbed there by the attenuator. Finally, the other half goes into the fourth arm to the detector, which has been left out of Fig. 5 for the sake of simplicity. At a maximum, therefore, 25% of the total energy can reach the detector, and is thus available for the reading. It can be shown that this energy loss is unavoidable without any special reference to the magic tee, which here acts as a "directional coupler" with a coupling factor of $\frac{1}{2}$. The directional coupling property of the magic tee[17] prevents the energy from going directly from the generator to the detector without passing through the resonator. Otherwise interference with the signal reflected by the resonator occurs. The voltage of the detector would be independent of the phase angle of reflection at the resonator. The oscillograph images would no longer have the form of resonance curves, but in some cases would look like "discriminator" curves. From the detector the rectified energy passes to a low frequency amplifier to which the cathode ray tube is connected. The Klystron is frequency modulated and the time base on the oscillograph runs synchronously with it. The spectrum emitted by the Klystron, including the absorption at the resonances of the cavity (see Fig. 6a) thus appears on the screen. If the time base is stretched out and the sign of the image is reversed, an impression of ordinary resonance curves is received (Fig. 6b).

If the coupling is weak, the height of curves A is proportional to the energy P_{ab}, absorbed by the resonator, independently of the characteristic of the detector. This is why the measurements for excitation statistics were made according to the absorption method:

$$(1-A)^{\beta} \sim P_{refl.} = P_{input} - P_{ab}.$$

Hence for $A \ll 1$ and P_{input} = const, it follows that $A \sim P_{ab}$. From the definition of the fluctuation square of the excitation

$$\Phi = \frac{\overline{a^2}}{\bar{a}^2} = \frac{\overline{P_{ab}}}{\overline{\sqrt{P_{ab}}}^2},$$

the measurement condition

$$\Phi = \frac{\frac{1}{N}\sum A}{\left(\frac{1}{N}\sum \sqrt{A}\right)^2}$$

then follows.

The measurement of the spacing statistics, of Ψ, is carried out basically in the same manner. However, the resonator is coupled on not by a diaphragm, but by a small antenna, the extension of the central conductor of a co-axial cable, projecting a small distance into the cavity. A bridge piece from hollow conductor to co-axial cable is then attached to the measuring arm of the magic tee. The shape and length of the antenna is adjusted by trial and error until all EF are covered as uniformly as possible. Fig. 7a is an example of a very uneven excitation. Result of this is that several weaker EF are masked by the more strongly excited EF. Fig. 7b shows the same place in the spectrum with better adjustment of the antenna.

For measurement of the spacing statistics, a frequency scale is also required on the screen of the cathode ray tube. Detailed data on such special oscillographs may be found in the microwave literature[18]. Here it need only be said that the frequency marks are produced by interference of the frequency modulated Klystron with an unmodulated one. The demodulated interference signal is fed to a detuned high frequency amplifier and after a second demodulation goes to the control grid of the cathode ray tube. There it produces two dark marks on the time base which are twice as far apart as the frequency selected for the high frequency amplifier. The marks are adjusted laterally by varying the frequency of the unmodulated Klystron. Thus every frequency difference appearing on the screen can be evaluated with the accuracy of the high frequency amplifier.

I should like to take this opportunity of thanking Professor Dr. E. Meyer for his generous support of the work and his many valuable suggestions. I also wish to thank the Deutsche Forshungsgemeinschaft for the provision of a stipendium.

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Table I

n^2	Degree of Degeneration Elec.	Acous.	n_x, n_y, n_z	Permutations	Polarizations.
150	30	15	11, 5, 2	6	2
			10, 7, 1	6	2
			10, 5, 5	3	2
151	0	0			
152	18	9	12, 2, 2	3	2
			10, 6, 4	6	2
153	36	21	12, 3, 0	6	1
			11, 4, 4	3	2
			10, 7, 2	6	2
			9, 6, 6	3	2
			8, 8, 5	3	2
154	24	12	12, 3, 1	6	2
			9, 8, 3	6	2
155	24	12	11, 5, 3	6	2
			9, 7, 5	6	2
156	0	0			
157	18	12	12, 3, 2	6	2
			11, 6, 0	6	1
158	24	12	11, 6, 1	6	2
			10, 7, 3	6	2
159	0	0			
160	6	6	12, 4, 0	6	1
161	48	24	12, 4, 1	6	2
			11, 6, 2	6	2
			10, 6, 5	6	2
			9, 8, 4	6	2
162	21	12	12, 3, 3	3	2
			11, 5, 4	6	2
			9, 9, 0	3	1
Su.	249	135			

Table II

n^2	Deg. of Degeneration	No. of EF Found	Division MHz	Distance to Next Group MHz
150	30	13	15	46
151	0	0		
152	18	9	8	20
153	36	15	12	15
154	24	14	17	17
155	24	17	11	50
156	0	0		
157	18	9	10	20
158	24	12	11	46
159	0	0		
160	6	6	6	18
161	48	22	19	13
162	21	15	14	

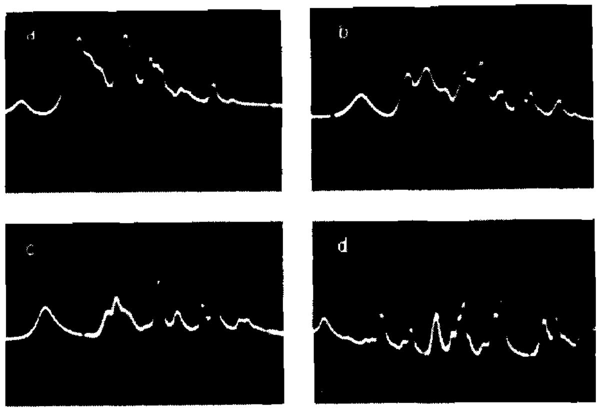

Fig. 1

The 24 eigen frequencies of the group $n^2 = 158$, 12 of which can be seen here. In all four oscillograms the distance between the dark marks denotes 15 MHz. In photograph (a) we see the group for the uncontracted cube. The separation due to mechanical irregularities is about 11 MHz. Photographs (b) to (d) show this group with progressive contraction of two opposite cube faces. The group spreads out progres- and in (d) has already joined up at the right with the group $n^2 = 157$. The contraction required to fill up this gap was only 0.25 mm. (Higher frequencies are plotted at the left).

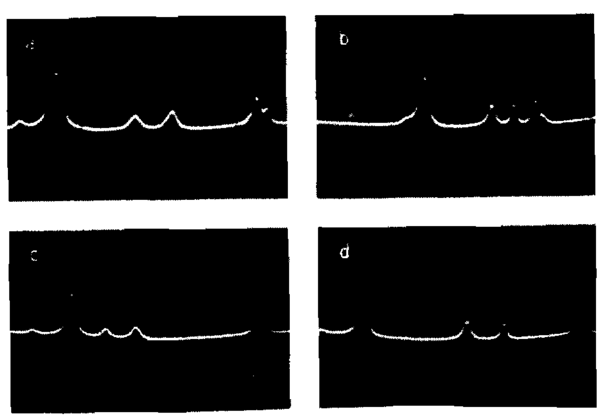

Fig. 2

The six eigen frequencies of the group $n^2 = 160$. The distance between the dark marks in this case signifies 5 MHz. The separation of the group is approximately 6 MHz. Photograph (a) shows the group in the uncontracted state. The oscillograms of (b) to (d) show the groups for small contractions in one of the three edge directions of the cube. Since this group involves quasi-axial waves, two of the six eigen frequencies in each case are displaced towards higher frequencies (to the left), while the remaining four remain relatively unchanged.

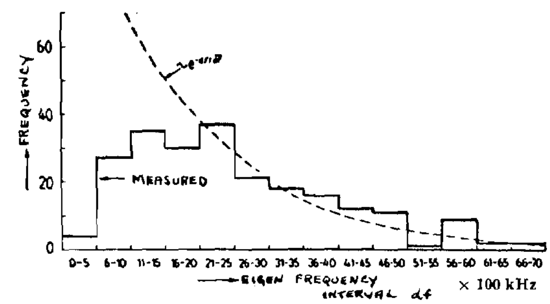

Fig. 3

Spacing statistics of the eigen frequencies of a near cube with dimensions 200 x 199 x 198 mm³ for a wave length of approximately 3.2 cm. The broken line is the theoretical frequency curve for the different intervals. The solid line represents the measurements. Good agreement is obtained for intervals above 2 MHz. Below 2 MHz the number of resonances found is continuously further below the theoretical number, owing to the limited resolving power.

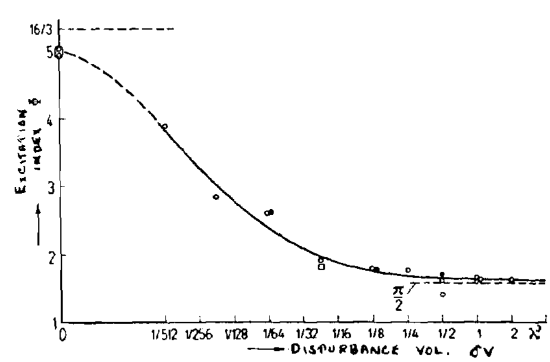

Fig. 4

The excitation statistics of the natural vibrations of a parallelepiped with dimensions 101 x 150 x 299 mm³ at a wavelength of approximately 3.2 cm. The excitation statistics are plotted against the disturbance volume. For the open circles the interference consists of small cubes placed on the floor of the space. The filled-in circles stand for measurements with small cones and the square for a measurement for which a cube was divided into 8 equal parts and distributed over the floor of the space. The lack of dependence of the excitation parameter Φ on the shape of the disturbing element is evident. It is striking that even for very small interference element volumes in comparison with the volume of the whole space, the excitation statistics are completely random, characterized by the value 1.57 for the excitation parameter.

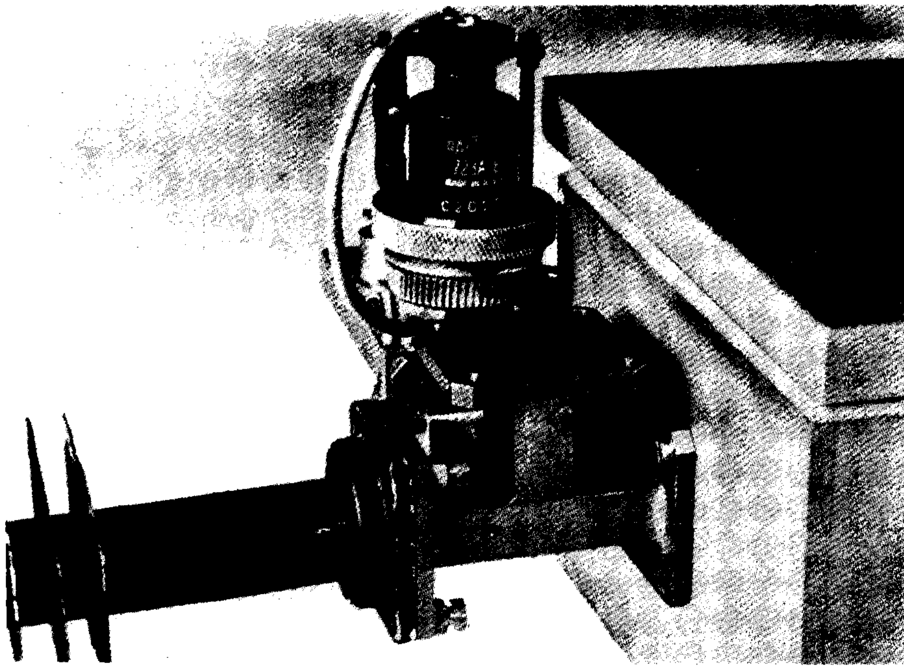

Fig. 5

The microwave part of the measuring apparatus. The Klystron generator for a 3 cm electric wave, the "magic tee", the non-reflecting absorber (left) and the cavity resonator being investigated (right) are shown

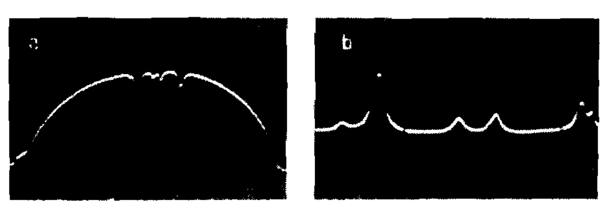

Fig. 6

Photograph (a) shows the spectrum of the Klystron generator with cavity resonator absorption points. In photograph (b) we see the centre part of the spectrum with extended frequency axis and reversed ordinate direction

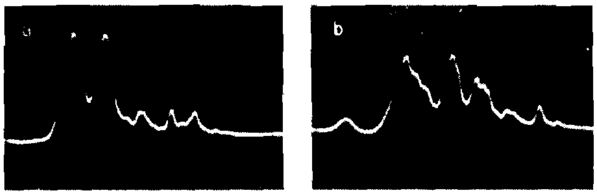

Fig. 7

A group of eigen frequencies with very different excitation of the individual eigen frequencies. In photograph (b) an adjustment of the "antenna" resulted in a more uniform excitation. This reduced the probability of concealing weakly excited eigen frequencies.