On the general solution of the basic equation of thermal conduction

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PREFACE

One of the projects of the Fire Section of the Division of Building Research concerns the development of a method for calculating the fire endurance of building elements. The main theoretical problem in this work is that the classical solutions of the Fourier equation, based on constancy of properties, cannot be used because of the wide variations in temperature involved in this experimental work.

Three papers by Masao Sawada that deal with problems of the heat conduction encountered when the heat capacity and thermal conductivity of the solid are variable, have been translated and issued as NRC Technical Translations Nos. 895, 896 and 897. From the work of Sawada it is seen that the more perfectly the mechanism of heat conduction within the solid is approached, the more limited the applicability of the method becomes, as far as the shape of the solid or the boundary conditions are concerned. Although the introduction of various numerical methods and computer techniques largely reduce the practical value of such analytical methods, there are certain fields where they are indispensable.

Ottawa,
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R.F. Legget,
Director

NATIONAL RESEARCH COUNCIL OF CANADA

Technical Translation 895

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Author:	Masao Sawada
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Translator:	K. Shimizu

ON THE GENERAL SOLUTION OF THE BASIC EQUATION OF THERMAL EQUATION

Abstract

The present article deals first with a solution of the general equation under the assumptions that heat capacity is a constant and that thermal conductivity is a function of temperature. It also deals with Van Dusen's solution in which thermal diffusivity and thermal conductivity are assumed to be a constant and a function of temperature, respectively. Further, it discusses the solution of the fundamental equation of bodies with varying conductivities in infinite or semi-infinite bodies and derives a solution for homogeneous bodies.

1. The Fundamental Equation of Thermal Conduction with Thermal Conductivity as a Function of Temperature

(A). The fundamental equation of heat flow in which specific heat, γ, density, ρ, or heat capacity, $\rho\gamma$, are assumed to be constant and in which thermal conductivity, λ, is assumed to be a function of temperature, takes the same form as the fundamental equation in which λ is assumed to be a ternary function. (More precisely, λ is a function of temperature and of three coordinates: here it is a function of temperature only.)

$$\rho\gamma\, \partial v/\partial t = \partial(\lambda \partial v/\partial x)/\partial x + \partial(\lambda\partial v/\partial y)/\partial y + \partial(\lambda\partial v/\partial z)/\partial z \qquad \cdots\cdots (1)$$

$$= 1/r\cdot\partial(\lambda r\partial v/\partial r)/\partial r + 1/r^2\cdot\partial(\lambda\partial v/\partial\theta)/\partial\theta + \partial(\lambda\partial v/\partial z)/\partial z \qquad \cdots\cdots (1_1)$$

$$= 1/r^2\partial(\lambda r^2\partial v/\partial r)/\partial r + 1/\overline{r^2\sin\theta}\,\partial(\lambda\sin\theta\partial v/\partial\theta)/\partial\theta + 1/\overline{r^2\sin^2\theta}\,\partial(\lambda\partial v/\partial\phi)/\partial\phi \qquad \cdots\cdots (1_2)$$

$$\lambda = f(v),\ v = f(x, y, z, t),\ d\lambda = (\partial\lambda/\partial v)\,dv,$$

$$dv = (\partial v/\partial x)\,dx + (\partial v/\partial y)dy + (\partial v/\partial z)\,dz + (\partial v/\partial t)\,dt.$$

(B). The fundamental equation, in which thermal diffusivity $\lambda/\rho\gamma = a_0^2$ is assumed to be a constant as in Van Dusen's thesis* and $\lambda = f(v)$, is still identical in form with (1), (1_1) and (1_2). Let $\int^v \lambda dv = u$,

$$a_0^{-2}\,\partial u/\partial t = \nabla^2 u \quad \cdots\cdots (1_3)$$

$$= 1/r\,\partial(r\partial u/\partial r)/\partial r + 1/r^2\partial^2 u/\partial^2\theta + \partial^2 u/\partial z^2 \quad \cdots\cdots (1_4)$$

$$= 1/r^2\partial(r^2\partial u/\partial r)/\partial r + 1/r^2\sin\theta\,\partial(\sin\theta\partial u/\partial\theta)/\partial\theta + 1/r^2\sin^2\theta\,\partial^2 u/\partial\phi^2 \quad \cdots\cdots (1_5)$$

i.e., the same form as the fundamental equation in which ρ, γ and λ are assumed to be constants.

(C). When the flow is constant, transform (A) by equating

$\partial v/\partial t = 0,\ \int^v \lambda dv = u$ or (B) by equating $\partial u/\partial t = 0$. Then

$$0 = \nabla^2 u \quad \cdots\cdots (1_6)$$

$$= 1/r\,\partial(r\partial u/\partial r)/\partial r + 1/r^2\partial^2 u/\partial\theta^2 + \partial^2 u/\partial z^2 \quad \cdots\cdots (1_7)$$

$$= 1/r^2\partial(r^2\partial u/\partial r)/\partial r + 1/r^2\sin\theta\,\partial(\sin\theta\,\partial u/\partial\theta)/\partial\theta + 1/r^2\sin^2\theta\,\partial^2 u/\partial\phi^2. \quad \cdots\cdots (1_8)$$

(D). Let the rate of radiation and the normal to the radiation surface be ε and N, respectively. Then

$$\partial u/\partial N = \partial\left[\int^v \lambda dv\right]/\partial N = \partial\left[\int^v \lambda dv\right]/\partial v \cdot \partial v/\partial N = \lambda\partial v/\partial N$$

i.e., the equation expresses the amount of heat flow through the surface. In the case of natural cooling:

$$\partial u/\partial N \pm \varepsilon v = \lambda\partial v/\partial N \pm \varepsilon v = 0 \cdots\cdots (1_9)$$

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* J. Soc. Mech. Engrs., Japan, 33 (161), Abstract 124, p. 329.

Further, in an adiabatic case:

$$\partial u/\partial N = \lambda \partial v/\partial N = 0 \qquad \because\ \partial v/\partial N = 0,\ \lambda \neq 0.$$

2. A Solution of the Fundamental Equation with Thermal Conductivity as a Function of Temperature

[1°] One dimensional fundamental equation with $\rho\gamma$ = constant and $\lambda = f(v)$, where m' = 0 (flat slab), m' = 1 (circular cylinder), and m' = 2 (sphere)

$$\rho\gamma\, \partial v/\partial t = r^{-m'}\, \partial(\lambda r^{m'}\, \partial v/\partial r)/\partial r \cdots\cdots(2)$$

(i) When the flow is constant

$$\partial(\lambda r^{m'}\, \partial v/\partial r)/\partial r = 0 \cdots\cdots\cdots\cdots(2_1)$$

$$\int \lambda dv = A\int r^{-m'}\, dr + B = Ar + B \cdots\cdots m' = 0;$$
$$= A \log r + B \cdots\cdots m' = 1;$$
$$= -A/r + B \cdots\cdots m' = 2.$$

If $\qquad \lambda = \lambda_0 + \lambda_1 v^n \qquad \int \lambda dv = \lambda_0 v + \lambda_1 v^{n+1}/\overline{n+1},$

and if $\qquad \lambda = \lambda_0 e^{nv} \qquad \int \lambda dv = \lambda_0 e^{nv}/n.$

Example:

If $\qquad \lambda = \lambda_0 + \lambda_1 v,\ v = v_a\ @\ r = r_a,\ v = v_i\ @\ r = r_i$

$$v = -\lambda_0/\lambda_1 + [(\lambda_0/\lambda_1 + v_i)^2 + (2\lambda_0/\lambda_1 + v_a + v_i)(v_a - v_i)f(r)]^{1/2}$$

where
$$f(r) = (x - x_i)/(x_a - x_i) \cdots\cdots \quad m' = 0;$$
$$= \log(r/r_i)/\log(r_a/r_i) \cdots\cdots m' = 1;$$
$$= (1/r_i - 1/r)/(1/r_i - 1/r_a) \cdots\cdots m' = 2.$$

Generally, since it is only necessary that $\lambda = f(v)$ be integrable, and an arbitrary function in seeking a temperature under a given boundary condition, there will be problems in which a a graphic solution is preferable to an algebraic solution.

Example:*

$$\lambda = 1{\cdot}5\ @\ v = 1{,}500^\circ\mathrm{C},\ \lambda = 0{\cdot}4\ @\ v = 100^\circ\mathrm{C}.$$

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* M. ten Bosch: Wärmeübertragung, pp. 53-54. The formulation of the equation on p. 53 is not correct.

With a flat slab 0.5 m in thickness,

$$\lambda = 0.3214 + 1.1/1{,}400 \cdot v$$

$$\therefore \quad v = -410 + 10\,(206.1 + 33{,}880\,x/0.5)^{1/2}$$

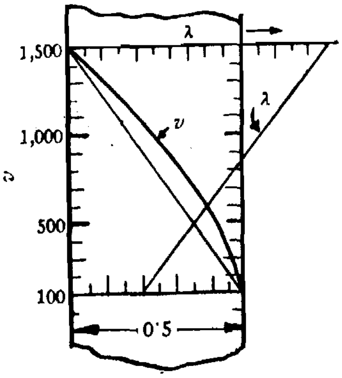

Fig. 1

(ii) When $\quad \partial v/\partial t \neq 0,\ m' = 0$

Let $\quad \xi = \pm i\alpha - i^k a_0^2\beta^2 t,\ a_0^2 = \lambda_0/\overline{\rho\gamma},\ i^k\beta^2 = \alpha^2,$

then:

$$d\xi = \pm i\alpha dx,\ d\xi = -i^k a_0^2\beta^2 dt.$$

From (2)

$$\lambda_0 \partial v/\partial\xi = \partial(\lambda\partial v/\partial\xi)/\partial\xi \cdots\cdots\cdots\cdots (2_2)^*$$

$$\lambda\partial v/\partial\xi = \lambda_0 v + c_0,\ \int \lambda/\overline{\lambda_0 v + c_0}\,dv = \xi + c_1$$

where $\quad c_0 = 0.$

If

$$\lambda = \lambda_0 + \lambda_1 v^n$$

$$\lambda \log v + \lambda_1/\lambda_0 \cdot v^n/n = \xi + c_1$$

$$\therefore \quad v\,e^{\lambda_1 v^n/\overline{n\lambda_0}} = A\exp.(\pm i\alpha x - i^k a_0^2\beta^2 t \cdots (2_3)$$

$$= \exp.(-a_0^2\beta^2 t)\,{}^{\cos}_{\sin}(\beta x),$$

$$k = 4 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2_4)$$

$$= \exp.(-\sqrt{1/2}\,\beta x)\,{}^{\cos}_{\sin}(-\sqrt{1/2}\,\beta x$$

$$+ a_0^2\beta^2 t),\ k = 3 \cdots\cdots\cdots\cdots (2_5)$$

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* If one lets $\partial v/\partial\xi = p$, $p = 0$ or $dp/dv + d\log\lambda/dv \cdot p = \lambda_0/\lambda$ i.e., $p = (\lambda_0 v + c_0)/\lambda$.

If λ is very small, expand the left side of (2_3) and take several terms. If $\lambda_1 = 0$, the solution reduces to the case of one dimensional rectangular coordinate in which $\lambda = \lambda_0$ (constant). Here $\exp(\lambda_1 v^n/n\lambda_0)$ is a correction term.

Next, if $\lambda = \lambda_0 e^{nv}$,

$$v \exp.\left[nv + \frac{n^2}{2\cdot 2!}v^2 + \frac{n^3}{3\cdot 3!}v^3 + \cdots\cdots\right]$$
$$= A\exp.(\pm i\alpha x - i^k a_0^2\beta^2 t) \quad \cdots\cdots(2'_3)$$

(iii) When $\partial v/\partial t \neq 0$; $m' = 0$, 1 and 2. If $\lambda = \lambda_0 \pm \lambda_1 v$ the substitutions $v = \mp \lambda_0/\lambda_1 - \rho\gamma/2(m' + 3)\lambda_1 \cdot r^2/t - \tau$, $\tau = 0$ or a constant. In (2) make both sides equal, and hence one obtains a type of particular solution.

[2°] Equation in rectangular coordinates

$$\rho\gamma\partial v/\partial t = \partial_x(\lambda\partial_x v) + \partial_y(\lambda\partial_y v) + \partial_z(\lambda\partial_z v)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots(2_6)$$

(i) Let

$$\text{(1)}, \quad \xi = i^{k_1}\alpha_1 x + i^{k_2}\alpha_2 y + i^{k_3}\alpha_3 z - i^k a_0^2\beta^2 t,$$
$$a_0^2 = \lambda_0/\rho\gamma, \quad -i^k\beta^2 = (i^{k_1}\alpha_1)^2 + (i^{k_2}\alpha_2)^2 + (i^{k_3}\alpha_3)^2,$$

then,

$$\lambda_0\partial v/\partial\xi = \partial(\lambda\partial v/\partial\xi)/\partial\xi.$$

Hence, if $\lambda = \lambda_0 + \lambda_1 v^n$

$$v\,e^{\lambda_1 v^n/n\lambda_0} = \exp.\xi \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2_7)$$

$$= e^{-a_0^2\beta^2 t}\,{\cos \atop \sin}(\alpha_1 x)\cdot{\cos \atop \sin}(\alpha_2 y)$$
$$\cdot{\cos \atop \sin}(\alpha_3 z), \quad k = 4 \cdots\cdots\cdots\cdots(2'_7)$$

$$= e^{-a_0^2\beta^2 t}\,{\cos \atop \sin}(\alpha_1 x)\cdot{\cosh \atop \sinh}(\alpha_2 y)$$
$$\cdot{\cosh \atop \sinh}(\alpha_3 z), \quad k = 4 \cdots\cdots\cdots(2''_7)$$

$$= \exp.[-\sqrt{1/2}\beta(\phi_1 x + \phi_2 y + \phi_3 z)]$$
$$\cdot{\cos \atop \sin}(-\sqrt{1/2}\beta\phi_1 x + a_0^2\phi_1^2\beta^2 t)$$
$$\cdot{\cos \atop \sin}(-\sqrt{1/2}\beta\phi_2 y + a_0^2\phi_2^2\beta^2 t)$$
$$\cdot{\cos \atop \sin}(-\sqrt{1/2}\beta\phi_3 z + a_0^2\phi_3^2\beta^2 t),$$
$$k = 3 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2'''_7)$$

When $\lambda = \lambda_0 e^{nr}$, in each case, the right side becomes identical with that of (2_7); however, the left side takes the form of $(2'_3)$.

If $\lambda = \lambda_0 \pm \lambda_1 v$, the following solution is merely a particular solution of (2_7).

$$v = \mp \lambda_0/\lambda_1 - \rho\gamma/10\,\lambda_1 [(x-x')^2 + (y-y')^2 + (z-z')^2]/\overline{t-\tau}\ .$$

(ii) When the flow is constant.

If we let $$u = \int^v \lambda dv,$$

then $$\nabla^2 u = 0$$

$$\therefore \int \lambda dv = \frac{\cos}{\sin}(\sqrt{\alpha^2+\beta^2}x)\cdot\frac{\cosh}{\sinh}(\alpha y)\cdot\frac{\cosh}{\sinh}(\beta z) \cdots\cdots\cdots\cdots (2_8)$$

Let $$u = Ax^2 + By^2 + Cz^2.$$

If $$A + B + C = 0,$$
then

$$\int \lambda dv = A(x-x')^2 + B(y-y')^2 + C(z-z')^2$$
$$= Ax^2 + A'x + By^2 + B'y + Cz^2 + C'z + D.$$

In two dimensions,

$$\int \lambda dv = \varphi_1(x+iy) + \varphi_2(x-iy)$$

[3°] When $\rho\gamma/\lambda = a_0^2,\ \lambda = f(v).$

$$\int \lambda dv = e^{-a_0^2\beta^2 t}\frac{\cos}{\sin}(\alpha_1 x)\cdot\frac{\cos}{\sin}(\alpha_2 y)\cdot\frac{\cos}{\sin}(\alpha_3 y);\ m'=0,\ k=4 \cdots\cdots (2_9)$$

$$= \exp.[-\sqrt{1/2}\beta(\phi_1 x + \phi_2 y + \phi_3 z)]$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{1/2}\phi_1\beta x + a_0^2\phi_1^2\beta^2 t)$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{1/2}\phi_2\beta y + a_0^2\phi_2^2\beta^2 t)$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{1/2}\phi_3\beta z + a_0^2\phi_3^2\beta^2 t),$$
$$m'=0,\ k=3 \cdots\cdots\cdots (2_{10})$$

$$= \exp.[-a_0^2(\alpha_1^2+\alpha_3^2)t]\frac{J_n}{Y_n}(\alpha_1 r)\cdot\frac{\cos}{\sin}(n\theta)\frac{\cosh}{\sinh}(\alpha_3 z),\ m'=1,\ k=4$$
$$\cdots\cdots\cdots (2_{11})$$

$$= \exp.(ia_0^2\alpha_1^2 t - \sqrt{\tfrac{1}{2}}\alpha_3 z)$$

$$\begin{bmatrix} I_n \\ K_n \end{bmatrix}(\sqrt{i}\,\alpha_1 r) \begin{matrix}\cos\\ \sin\end{matrix}(u\theta)$$

$$\cdot \begin{matrix}\cos\\ \sin\end{matrix}(-\sqrt{\tfrac{1}{2}}\alpha_3 z + a_0^2\alpha_3^2 t);$$

$$m' = 1,\ k = 3 \quad \cdots\cdots\cdots\cdots (2_{12})$$

$$= \exp.(-a_0^2\alpha^2 t)\left[r^{-1/2} \begin{matrix} J_{n+1/2} \\ J_{-n-1/2}\end{matrix}(\alpha r)\right]$$

$$\cdot\left[(1-\mu^2)^{m/2} \begin{matrix} P_n^{(m)} \\ Q_n^{(m)}\end{matrix}(\mu)\right]\left[\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)\right];$$

$$m' = 2,\ k = 4 \quad \cdots\cdots\cdots\cdots (2_{13})$$

$$= \exp.(ia_0^2\alpha^2 t)\left[r^{-1/2} \begin{matrix} I_{n+1/2} \\ I_{-n-1/2}\end{matrix}(\sqrt{i}\,\alpha r)\right]$$

$$\cdot\left[(1-\mu^2)^{m/2} \begin{matrix} P_n^{(m)} \\ Q_n^{(m)}\end{matrix}(\mu)\right]\left[\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)\right];$$

$$m' = 2,\ k = 3 \quad \cdots\cdots\cdots\cdots (2_{14})$$

where m and n are integers, and $\mu = \cos\theta$.

(ii) When $\partial v/\partial t = 0$, $m' = 0$.

Let $\xi = \pm i\alpha - i^k a_0^2\beta^2 t,\ a_0^2 = \lambda_0/\rho\gamma,\ i^k/\beta^2 = \alpha^2$

$$d\xi = \pm i\alpha dx, \quad d\xi = -i^k a_0^2\beta^2 dt.$$

From (2)

$$\lambda_0 \partial v/\partial\xi = \partial(\lambda\partial v/\partial\xi)/\partial\xi \quad \cdots\cdots\cdots\cdots (2_2^{*})$$

$$\lambda\partial v/\partial\xi = \lambda_0 v + c_0, \quad \int \lambda/\overline{\lambda_0 v + c_0}\, dz = \xi + c_1$$

where $c_0 = 0$.

If $\lambda = \lambda_0 + \lambda_1 v^n$

$$\lambda \log v + \lambda_1/\lambda_0 \cdot v^n/_n = \xi + c_1$$

$$\therefore\ v e^{\lambda_1 v^n/n\lambda_0} = A\exp.(\pm i\alpha x - i^k a_0^2\beta^2 t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots (2_3)$$

$$= \exp.(-a_0^2\beta^2 t)\begin{matrix}\cos\\ \sin\end{matrix}(\beta x),$$

$$k = 4 \quad \cdots\cdots\cdots\cdots\cdots\cdots (2_4)$$

$$= \exp.(-\sqrt{\tfrac{1}{2}}\beta x)$$

$$\cdot\begin{matrix}\cos\\ \sin\end{matrix}(-\sqrt{\tfrac{1}{2}}\,\beta x + a_0^2\beta^2 t),$$

$$k = 3 \quad \cdots\cdots\cdots\cdots\cdots\cdots (2_5)$$

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* If one lets $\partial v/\partial\xi = p$, $p = 0$ or $dp/dv + d\log\lambda/dv \cdot p = \lambda_0\lambda$, i.e., $p = (\lambda_{0v} + c_0)/\lambda$.

If λ_1 is very small, expand the left side of (2_3) and take several terms. If $\lambda_1 = 0$, it reduces to the one dimensional solution in rectangular coordinates when the conductivity is a constant.

If $\lambda = \lambda_0 e^{nv}$,

$$v \exp.\left[nv + \frac{n^2}{2\cdot 2!}v^2 + \frac{n^3}{3\cdot 3!}v^3 + \cdots\cdots\right] = A\exp.(\pm i\alpha x - i^k a_0^2\beta^2 t) \cdots\cdots(2'_3)$$

(iii) When $\partial v/\partial t \neq 0$, $m' = 0$, 1 and 2.

If $\lambda = \lambda_0 \pm \lambda_1 v$, let $v = T \mp \lambda_0/\lambda_1$.

Substituting in (2), we obtain

$$\rho\gamma/\lambda_1\, \partial T/\partial t = 1/r^{m'}\, \partial(Tr^{m'}\,\partial T/\partial r)/\partial r \cdots(2_4)$$

When $T = -\rho\gamma/\overline{2(m'+3)\lambda_1}\cdot r^2/t$ is substituted in (2_4), both sides of the equation become equal. Hence, the following solution is a type of particular solution, but not the required solution.

$$\begin{aligned} v &= \mp\lambda_0/\lambda_1 - \rho\gamma/\overline{6\lambda_1}\,(x-x')^2/\overline{t-\tau} \cdots\cdots m'=0 \\ &= \mp\lambda_0/\lambda_1 - \rho\gamma/\overline{8\lambda_1}\cdot r^2/\overline{t-\tau} \cdots\cdots\cdots\cdots m'=1 \\ &= \mp\lambda_0/\lambda_1 - \rho\gamma/\overline{10\lambda_1}\cdot r^2/\overline{t-\tau} \cdots\cdots\cdots\cdots m'=2 \end{aligned}$$

[2°] Equation in rectangular coordinates.

$$\rho\gamma\,\partial v/\partial t = \vartheta_x(\lambda\vartheta_x v) + \vartheta_y(\lambda\vartheta_y v) + \vartheta_z(\lambda\vartheta_z) \cdots\cdots\cdots\cdots\cdots\cdots(2_5)$$

(i) If we let

$$\xi = i^{k_1}\alpha_1 x + i^{k_2}\alpha_2 y + i^{k_3}\alpha_3 z - i^k a_0^2\beta^2 t,$$
$$a_0^2 = \lambda_0/\overline{\rho\gamma}, \quad -i^k\beta^2 = (i^{k_1}\alpha_1)^2 + (i^{k_2}\alpha_2)^2 + (i^{k_3}\alpha_3)^2$$

we obtain from (2_5) the expression (2_2).

Hence, if $\lambda = \lambda_0 + \lambda_1 v^n$

$$\begin{aligned} v e^{\lambda_1 v^n/\overline{n\lambda_0}} &= \exp.\xi \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2_6) \\ &= e^{-a_0^2\beta^2 t}\,{\cos \atop \sin}(\alpha_1 x)\,{\cos \atop \sin}(\alpha_2 y) \\ &\quad\cdot{\cos \atop \sin}(\alpha_3 z),\ k=4 \\ &= e^{-a_0^2\beta^2 t}\,{\cosh \atop \sinh}(\alpha_1 x)\,{\cosh \atop \sinh}(\alpha_2 y) \\ &\quad\cdot{\cosh \atop \sinh}(\alpha_3 z),\ k=4 \\ &= \exp.[-\sqrt{\tfrac{1}{2}}\beta(\phi_1 x + \phi_2 y + \phi_3 z)] \end{aligned}$$

$$\cdot\frac{\cos}{\sin}(-\sqrt{\tfrac{1}{2}}\beta\phi_1 x + a_0^2\phi_1^2\beta^2 t)$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{\tfrac{1}{2}}\beta\phi_2 y + a_0^2\phi_2^2\beta^2 t)$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{\tfrac{1}{2}}\beta\phi_3 z + a_0^2\phi_3^2\beta^2 t),$$
$$k=3$$

When $\lambda = \lambda_0 e^{nv}$, in each case, the right side is identical, but the left side takes the form of $(2'_3)$. If $\lambda = \lambda_0 + \lambda_1 v$, $v = T - \lambda_0/\lambda_1$, and $\lambda = \lambda_0 T$; hence

$$2\rho\gamma/\lambda_1 \,\partial T/\partial t = \nabla^2 T^2. \qquad (2_7)$$

Now, if we let $T = -a_1^2(x^2 + y^2 + z^2)/t$, $a_1^2 = \rho\gamma/\overline{m\lambda_1}$, $m = 2(m' + 3)$, $m' = 0$, 1 and 2, and substitute them in (2_7), both sides become equal in one , two and three dimensional cases. However, the following is merely a particular solution.

$$v = -\rho\gamma/\overline{10\lambda_1}\,[(x-x')^2 + (y-y')^2 + (z-z')^2]/\overline{t-\tau} - \lambda_0/\lambda_1 \quad \cdots\cdots\cdots(2_8)$$

(ii) When the flow is constant.

If we let

$$u = \int^v \lambda dv, \qquad \nabla^2 u = 0:$$

hence

$$\int \lambda dv = \frac{\cos}{\sin}(\sqrt{\alpha^2+\beta^2}x)\frac{\cosh}{\sinh}(\alpha y)$$
$$\cdot\frac{\cosh}{\sinh}(\beta z) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2_9)$$

Next, let $u = Ax^2 + By^2 + Cz^2$. If $A + B + C = 0$

$$\int \lambda dv = A(x-x')^2 + B(y-y')^2 + C(z-z')^2$$
$$= Ax^2 + A'x + By^2 + B'y + Cz^2 + C'z + D$$

In two dimensions

$$\int \lambda dv = \varphi_1(x+iy) + \varphi_2(x-iy)$$

[3°] When $\rho\gamma/\lambda = a_0^2 =$ constant and $\lambda = f(v)$

$$\int \lambda dv = e^{-a_0^2\beta^2 t}\frac{\cos}{\sin}(\alpha_1 x)\frac{\cos}{\sin}(\alpha_2 y)$$
$$\cdot\frac{\cos}{\sin}(\alpha_3 z);\ m'=0,\ k=4$$
$$= \exp.[-\sqrt{\tfrac{1}{2}}\beta(\phi_1 x + \phi_2 y + \phi_3 z)]$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{\tfrac{1}{2}}\phi_1\beta x + a_0^2\phi_1^2\beta^2 t)$$
$$\cdot\frac{\cos}{\sin}(-\sqrt{\tfrac{1}{2}}\phi_2\beta y + a_0^2\phi_2^2\beta^2 t)$$

$$\begin{matrix}\cos \\ \sin\end{matrix}(-\sqrt{\tfrac{1}{2}}\phi_3\beta z + a_0^2\phi_3^2\beta^2 t);$$

$m' = 0,\ k = 3$

$$= e^{ia_0^2\alpha_1^2 t - \sqrt{1/2}\alpha_3 z}\left[\begin{matrix}In \\ Kn\end{matrix}(\sqrt{i}\,\alpha_1 r)\right]$$

$$\cdot \begin{matrix}\cos \\ \sin\end{matrix}(n\theta)\begin{matrix}\cos \\ \sin\end{matrix}(-\sqrt{\tfrac{1}{2}}\alpha_3 z + a_0^2\alpha_3^2 t);$$

$m' = 1,\ k = 3$

$$= e^{-a_0^2\alpha^2 t}\left[r^{-1/2}\begin{matrix}J_{n+1/2} \\ J_{-n-1/2}\end{matrix}(\alpha r)\right]$$

$$\cdot\left[(1-\mu^2)^{m/2}\begin{matrix}P_n^{(m)} \\ Q_n^{(m)}\end{matrix}(\mu)\right]\left[\begin{matrix}\cos \\ \sin\end{matrix}(m\phi)\right];$$

$m' = 2,\ k = 4$

$$= e^{ia_0^2\alpha^2 t}\left[r^{-1/2}\begin{matrix}I_{n+1/2} \\ I_{-n-1/2}\end{matrix}(\sqrt{i}\,\alpha r)\right]$$

$$\cdot\left[(1-\mu^2)^{m/2}\begin{matrix}P_n^{(m)} \\ Q_n^{(m)}\end{matrix}(\mu)\right]\left[\begin{matrix}\cos \\ \sin\end{matrix}(m\phi)\right];$$

$m' = 2,\ k = 3$

where m and n are integers $u = \cos\theta$.

3. Alternate Solution to Heat Flow Equation in Bodies with Varying Conductivities

In solving the fundamental heat flow equation in homogeneous bodies in one dimension, trignometric function is used in the case of a flat slab, cylindrical function in the case of a circular cylinder, and double trignometric or cylindrical function for a sphere. In problems of constant heat flow algebraic functions are used. Further, in bodies with varying conductivities, if the varying conductivities can be expressed in a simple power series, a solution can be obtained in the form of a double cylindrical function. Even in this case, if the flow is constant, a solution can be obtained in many problems in the form of an algebraic, a logarithmic, or a trignometric function. In the following sections, the writer wishes to cite examples of solutions in finite bodies by the use of a double cylindrical function, and to derive solutions for infinite or semi-infinite bodies.

[1°] $\lambda = \lambda_0 r^{\mu}$, $\rho\gamma = \rho_0\gamma_0 r^{\mu'}$, $\lambda_0/\overline{\rho_0\gamma_0} = a_0^2$, Let $m' = 0$, 1 and 2, $\mu' - \mu = p - 2$, $m' + \mu - 1 = m$, $m/p = n$, and Z represent a cylinderical function. Then:

$$\rho\gamma\, \partial v/\partial t = r^{-m'}\partial(\lambda r^{m'}\,\partial v/\partial r)/\partial r \cdots\cdots (3)$$

Solution:

$$v = \exp.(-i^k a_0^2 p^2 \alpha^2 t)[Zn(2\alpha\sqrt{i^k r^p})]/\sqrt{r^m} \cdots\cdots\cdots\cdots\cdots\cdots (3_1)$$

(i) Next, let

$$\underline{r} = -\lambda_0 \int^r (\lambda r^{m'})^{-1}\,dr = r^{-m}/m;$$
$$d\underline{r} = -r^{-\mu-m'}\,dr, \quad \mu + m' \neq 1, \quad a^2 = a_0^2 m^{2+1/n},$$
$$\underline{t} = a^2 t; \quad (\rho\gamma/\lambda_0)\, r^{2m'+\mu} = a^{-2}\underline{r}^q, \quad q = -2 - 1/n.$$

Then, from (3):

$$\underline{r}^q\, \partial v/\partial \underline{t} = \partial^2 v/\partial \underline{r}^2 \cdots\cdots\cdots\cdots\cdots (3_2)$$

$$\therefore\ v = \exp.(-i^k n^{-2}\alpha^2 \underline{t})[Zn(2\alpha\sqrt{i^k \underline{r}^{-1/n}})]\sqrt{\underline{r}} \cdots\cdots\cdots\cdots\cdots\cdots (3_3)$$

$$= \exp.(-i^k a_0^2 p^2 \beta^2 t)[Zn(2\beta\sqrt{i^k r^p})]/\sqrt{r^m},$$
$$\beta^2 = m^{1/n}\alpha^2 \cdots\cdots\cdots\cdots\cdots\cdots (3_4)$$

(ii) When $v = \phi(\underline{t})\exp[\psi(\underline{r})/\underline{t}]$ is substituted in (3_2), the following solution is obtained, in which r = 0 or a constant and A = 4π:

$$v = [A(\underline{t} - \tau)]^{-(1+q)/(2+q)}$$
$$\exp.[-\underline{r}^{2+q}/(2+q)^2(\underline{t} - \tau)] \cdots\cdots (3_5)$$

$$= [A(\underline{t} - \tau)]^{-(1+n)}$$
$$\cdot\exp.[-r^p/p^2(\underline{t} - \tau)] \cdots\cdots\cdots\cdots\cdots\cdots (3_6)$$

$$= [A(\underline{t} - \tau)]^{-1/2}\exp.[-\underline{r}^2/4(\underline{t} - \tau)];$$
$$q = 0, \quad n = -1/2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3_7)$$

When $m' = 0$, let $\lambda = \lambda_0(c \pm x)^{\pm\mu}$, and $\rho\gamma = \rho_0\gamma_0(c \pm x)^{\pm\mu'}$

(iii) In (i) if $\mu + m' = 1$, let

$$\underline{r} = \lambda_0\int^r (\lambda_0 r)^{-1}\,dr = \lambda \log r, \quad d\underline{r} = r^{-1}\,dr, \quad \underline{t} = a_0^2 t$$

then　とおくと $q = \mu' + m' + 1 = p$

$$e^{q\underline{r}}\, \partial v/\partial \underline{t} = \partial^2 v/\partial \underline{r}^2 \cdots\cdots\cdots\cdots\cdots (3_8)$$

$$\therefore\ v = (A\underline{t})^{-1}\exp.(-e^{q\underline{r}}/q^2\underline{t})$$
$$= (Aa_0^2 t)^{-1}\exp.(-r^p/a_0^2 p^2 t) \cdots\cdots\cdots (3_9)$$
$$= (A\underline{t})^{-1/2}\exp.(-\underline{r}^2/4\underline{t}) \qquad q = 0,$$

Solution: See (3_5). Or

$$(Aa_0^2t)^{-1/2}\exp.[-(\log r)^2/4a_0^2t]\cdots(3_{10})$$

(iv) In the case of a homogeneous body:

$$\lambda_0/\rho_0\gamma_0 = a_0^2,\ A = 4\pi.$$

From equations (3), (3_2), and (3_8), and from solutions (3_5) and (3_9) the following solution is easily obtained:

Flat slab $\quad v=(Aa_0^2t)^{-1/2}\exp.[-(x-x')^2/4a_0^2t]$ $\cdots\cdots\cdots\cdots(3_{11}^*)$

Circular cylinder $\quad =(A\underline{t})^{-1}\exp.[-e^{2z}/4\underline{t}]$

$$=(Aa_0^2t)^{-1}\exp.(-r^2/4a_0^2t)\cdots(3_{12}^*)$$

Sphere $\quad =(A\underline{t})^{-3/2}\exp.[-\underline{r}^{-2}/4\underline{t}]$

$$=(Aa_0^2t)^{-3/2}\exp.(-r^2/4a_0^2t)\cdots(3_{13}^*)$$

[2°] When $\lambda=\lambda_0e^{mx}$, $\rho\gamma=\rho_0\gamma_0e^{\mu' x}$, $m'=0$.

Let $\quad a_0^2=\lambda_0/\rho_0\gamma_0,\ \ \mu'-m=p,\ \ m/p=n,\ \ e^x=y,$

$z=y^{-m}/m,\ \ a^2=a_0^2m^{-q},\ \ q=-2-1/n,$

$\underline{t}=a^2t$.

Then, from (3)

$$a_0^{-2}e^{\mu' x}\partial v/\partial t=\partial(e^{mx}\partial v/\partial x)/\partial x$$

or $\quad a_0^{-2}y^{\mu'-1}\partial v/\partial t=\partial(y^{1+m}\partial v/\partial y)/\partial y$

or $\quad z^q\partial v/\partial t=\partial^2v/\partial z^2\quad$ the same form as (3_5).

$$\therefore\ v=\exp.(-i^ka_0^2p^2\alpha^2t)$$

$$[Zn(2\alpha\sqrt{i^ke^{px}})]/\sqrt{e^{mx}}\cdots\cdots\cdots\cdots(3_{14})$$

$$=(A'a^2t)^{-(1+n)}\exp.(-e^{px}/a_0^2p^2t),$$

$$A'=4\pi a_0^2/a^2\cdots\cdots\cdots\cdots\cdots\cdots(3_{15})$$

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* Carslaw: Conduction of Heat, pp. 29, 30, 150 and 184.

[3°] When $\lambda = f(v)$, $\lambda/\rho\gamma = a_0^2$ = constant; from (3_{11}), (3_{12}) and (3_{13}):

$$\int \lambda dv = (4\pi a_0^2 t)^{-1/2} \exp.[-(x-x')^2/4a_0^2 t],$$

$$m' = 0 \cdots\cdots (3_{16})$$

$$= (4\pi a_0^2 t)^{-1} \exp.[-r^2/4a_0^2 t], \quad m' = 1 \cdots\cdots (3_{17})$$

$$= (4\pi a_0^2 t)^{-3/2} \exp.[-r^2/4a_0^2 t], \quad m' = 2 \cdots\cdots (3_{18})$$

Remarks: Solutions have been worked out theoretically by many physicists on problems of heat conduction and of instantaneous heat source in homogeneous bodies of infinite ($-\infty \to +\infty$) and semi-infinite ($0 \to +\infty$) length*. In recent years, the works of Tamura** and Yamagata*** may be cited in connection with the former field.

Conclusions

(1) While a ternary solution in rectangular, cylindrical and in polar coordinates is possible as in the case of a homogeneous body, when thermal conductivity is a constant and thermal diffusivity is a function of temperature, a solution of the fundamental equation, in which thermal capacity is constant and thermal conductivity is a function of temperature, is limited generally to that in rectangular coordinates except when the flow is constant.

(2) As a solution of the fundamental equation of bodies with varying conductivities, we can obtain an ordinary solution (as finite bodies). In addition we can also obtain solutions for homogeneous infinite and semi-infinite bodies.

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* Carslaw: Conduction of Heat, pp. 29, 30, 150 and 184.

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