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# NOTE ON THE INDEPENDENT SAMPLING OF MEAN-SQUARE PRESSURE IN REVERBERANT SOUND FIELDS

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RÉSUMÉ

On traite dans le présent document du concept de prélèvement d'échantillons indépendants de la pression quadratique moyenne dans les champs acoustiques réverbérés dus à des sons purs et on déduit les résultats de Lubman sur la fonction de covariance de la pression quadratique moyenne.

# Note on the independent sampling of mean-square pressure in reverberant sound fields

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A discussion of the concept of independent sampling of mean-square pressure in pure-tone reverberant sound fields is presented together with a derivation of Lubman's result on the covariance function of the mean-square pressure.

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## INTRODUCTION

The need for spatial averaging of the mean-square pressure $\overline{p^2}$ in connection with laboratory measurements of sound power and sound transmission loss in reverberation rooms is well known. Several papers[1–8] in the *Journal* have been devoted to discussion of various types of spatial averaging. The concept of independent sampling can be accepted readily for such incoherent reverberant sound fields as are created by narrow-band random noise excitation. For pure-tone excitation, however, the reverberant sound fields created in rooms of fixed geometry are coherent[9] and the applicability of this concept needs to be clarified. The main objective of this Note is to present a more consistent picture of the subject and to provide a derivation of Lubman's result on the spatial covariance of the mean-square pressure that is essential to many of the previous discussions and to the present one.

## I. REVERBERANT FIELD UNDER PURE-TONE EXCITATION

Rooms with fixed geometry will be considered. At frequencies above the Schroeder's large room frequency,[10]

$$f_c = 2000(T_{60}/V)^{1/2}, \tag{1}$$

where $T_{60}$ is the reverberation time ($s$) and $V$, room volume ($\mathrm{m}^3$), a large number of room modes will be excited to vibrate at the source frequency. Using the free-wave model,[4] the reverberant sound field can be idealized as consisting of a large number of plane waves with equal amplitudes but different phases. At any point in the room sufficiently far from the source and any reflecting surfaces the directions of propagation of the plane-wave trains are assumed to be uniformly distributed in all directions. The acoustic pressure at such a location can be written as

$$p(\mathbf{r}, t) = \sum_{i=1}^{N} \cos[\omega t + \phi_i(\mathbf{r})], \tag{2}$$

where $\phi_i$ is the phase of the $i$th plane wave and $\omega$ the driving frequency of the source. $\{\phi_i\}$ is considered to be a set of $N$ statistically independent random variables uniformly distributed on $(0, 2\pi)$. Note that Eq. (2) can be rewritten as

$$p(\mathbf{r}, t) = A\cos(\omega t + \alpha), \tag{3}$$

where

$$A^2 = \left(\sum \cos\phi_i\right)^2 + \left(\sum \sin\phi_i\right)^2,$$

$$\alpha = \tan^{-1}\left(\sum \sin\phi_i / \sum \cos\phi_i\right).$$

In other words, the sound pressures at all points in space vary sinusoidally in time, but with different amplitudes and phases. As a result, the temporal mean-square pressure $\overline{p^2}$ shows large variation with position. For a practical estimate of the space-averaged value of $\overline{p^2}$ it is advisable to obtain a finite number of independent samples of $\overline{p^2}$. To ensure independence, a necessary condition is that the spatial covariance between paired samples is negligibly small. As the estimation process involves spatial averaging of $\overline{p^2}$, it is appropriate to assume that the random process describing the field is stationary and ergodic with respect to position. Thus ensemble average can be replaced by spatial average and different statistics of the field are independent of position and orientation. The spatial covariance of the mean-square pressure can be defined as

$$\begin{aligned} R(r) &= \mathrm{cov}[\,\overline{p^2}(\mathbf{r}_1),\ \overline{p^2}(\mathbf{r}_2)] \\ &= \langle \overline{p^2}(\mathbf{r}_1)\,\overline{p^2}(\mathbf{r}_2)\rangle - \langle \overline{p^2}\rangle^2, \end{aligned} \tag{4}$$

where $\langle\ \rangle$ denotes spatial averaging, $r = |\mathbf{r}_2 - \mathbf{r}_1|$, and $\langle \overline{p^2}\rangle$ represents the space-averaged value of $\overline{p^2}$.

Lubman[2] presented a result for the normalized covariance function without proof. Only recently Jacobsen[11] provided a derivation of Lubman's result for a pure-tone field using Hilbert transform technique. A more conventional approach is possible, however, and will be outlined in Sec. II. First, it is necessary to summarize several second-order statistics of this field.

### A. Probability density of mean-square pressure

Using the free-wave model mentioned in the previous section, Waterhouse[4] has derived the probability density function for the normalized mean-square pressure. If $I$ is defined by

$$I = \overline{p^2}(\mathbf{r})/\langle \overline{p^2}\rangle, \tag{5}$$

the probability density function $P(I)$ is

$$P(I) = e^{-I}. \tag{6}$$

### B. Joint probability density of instantaneous pressure

In Eq. (2), if $N$ is large, it follows from the central-limit theorem[12] that the instantaneous pressure $p$ is Gaussian distributed for any specific time, $t = t_i$. For a pair of points that

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are far enough apart to be uncorrelated their joint probability density is the product of two Gaussian density functions. For other pairs of correlated points it is possible that their joint probability density may not be bivariate Gaussian, although they are Gaussian distributed individually.[13]

To ensure that the joint probability density is bivariate Gaussian a computer simulation of the reverberant field has been performed using the free-wave model. If the assumption that the directions of the plane waves are uniformly distributed is incorporated into Eq. (2), the instantaneous pressures at two adjacent locations can be written as, following Jacobsen's work,[11]

$$p(\mathbf{r}_1, t_1) = \sum_{j=1}^{M} \sum_{i=1}^{L} \cos[\omega t_1 + \phi_i(\mathbf{r}_1, j)]$$
$$p(\mathbf{r}_2, t_2) = \sum_{j=1}^{M} \sum_{i=1}^{L} \cos\left[\omega t_2 + \phi_i(\mathbf{r}_1, j) - kr\cos\left(\frac{j\pi}{M}\right)\right], \tag{7}$$

where $L =$ positive integer value of $(N\pi/2M)\sin(j\pi/M)$, $r = |\mathbf{r}_2 - \mathbf{r}_1|$, and $k$ is the wavenumber ($=\omega$/speed of sound). To generate Eq. (7), the zenith angle ($0 \leqslant \theta \leqslant \pi$) has been divided into $M$ equal increments. The factor $(N\pi/2M)\sin(j\pi/M)$ represents the number of incident plane waves distributed over the ring element, as shown in Fig. 1. The spatial variation of the amplitude of the pair of points at any instant of time was simulated on the computer by generating different sets of $\phi_i(\mathbf{r}_1, j)$ using a routine that produces uniformly distributed random numbers.

It is rather difficult to test the normality of a two-dimensional joint-probability density function. A simpler approach is to rely on the fact that two random variables having a bivariate Gaussian probability density function can be transformed into two independent Gaussian random variables by a rotation of coordinates (see, for example, Davenport and Root[14]) and to test the normality of probability density functions of the transformed variables individually. Such an approach was taken by the author in his computer simulation. A total of 2500 pairs of points were generated for the probability density computation. A representative comparison of the density functions of the transformed variables with the standard Gaussian probability density is shown in Fig. 2. A chi-square test applied to these density functions using 27 class intervals of equal width showed that the hypothesis of normality was acceptable at the 0.05 level of significance. It is therefore concluded that the joint probability density is bivariate Gaussian for any pair of points at any instant of time in the free-wave model of a pure-tone reverberant sound field.

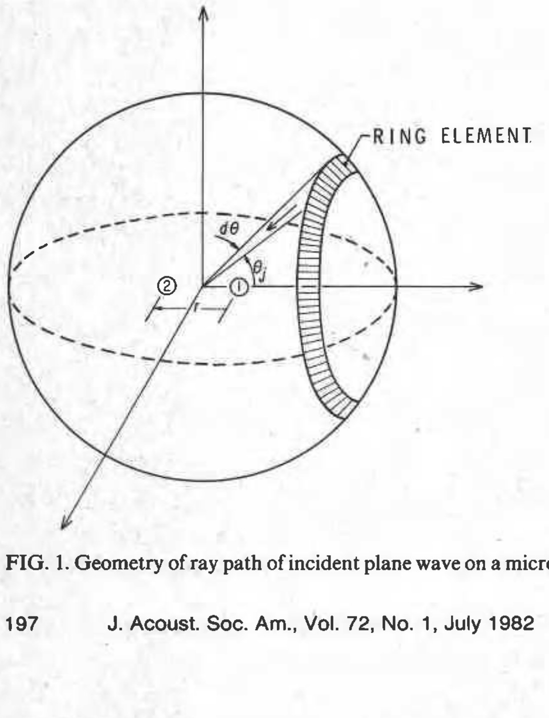


FIG. 1. Geometry of ray path of incident plane wave on a microphone pair.

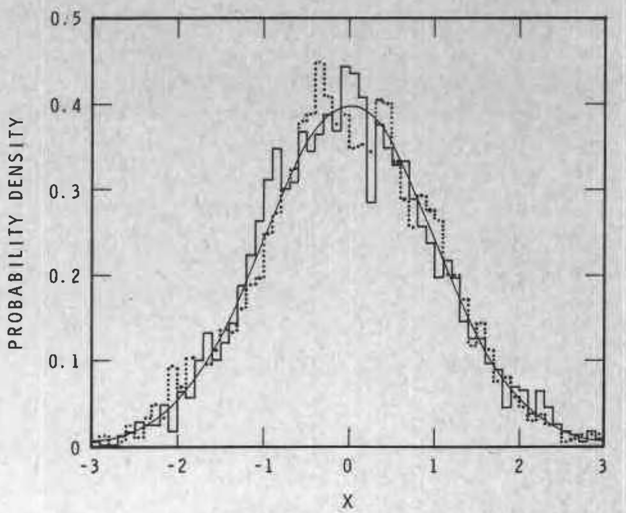


FIG. 2. Comparison of standardized probability density functions of transformed variables with Gaussian function. Combination of 30 waves with equal amplitudes but random phases. 2500 pairs of points were used.

### C. Space-time cross correlation of pressure

Using Eq. (7), the space-time cross correlation of pressure can be written as

$$\begin{aligned}\langle p(\mathbf{r}_1, t_1) p(\mathbf{r}_2, t_2)\rangle &= \sum_{m=1}^{M} \sum_{j=1}^{M} \sum_{l=1}^{L} \sum_{i=1}^{L} \langle \cos[\omega t_1 + \phi_l(\mathbf{r}_1, m)] \\ &\quad \times \cos[\omega t_2 + \phi_i(\mathbf{r}_1, j) - kr\cos(j\pi/M)]\rangle. \end{aligned} \tag{8}$$

It can be demonstrated that the main contribution to the quadruple summation comes from terms with $m = j$ and $l = i$. Hence

$$\begin{aligned}\langle p(\mathbf{r}_1, t_1) p(\mathbf{r}_2, t_2)\rangle &= \frac{1}{2} \sum_{j=1}^{M} \sum_{i=1}^{L} \cos\left[\omega(t_1 - t_2) + kr\cos\left(\frac{j\pi}{M}\right)\right] \\ &= \frac{1}{2} \sum_{j=1}^{M} \left(\frac{N\pi}{2M}\right) \left|\sin\left(\frac{j\pi}{M}\right)\right| \cos\left[\omega(t_1 - t_2) + kr\cos\left(\frac{j\pi}{M}\right)\right].\end{aligned}$$

Converting the summation over discrete zenith angles back to continuous integration,

$$\begin{aligned}\langle p(\mathbf{r}_1, t_1) p(\mathbf{r}_2, t_2)\rangle &= \left(\frac{N}{4}\right) \int_0^{\pi} \cos[\omega(t_1 - t_2) + kr\cos\theta]\sin\theta\, d\theta \\ &= (N/2)[\sin(kr)/kr]\cos[\omega(t_1 - t_2)]. \end{aligned} \tag{9}$$

The procedure is very similar to that given by Jacobsen,[11] but the interpretation is different. Although the normalized correlation of Eq. (9) with zero-time difference has the same form as that given by Cook *et al.*,[15] it has minimal practical value. Equation (9) only serves as an intermediate step to the final solution of the covariance of the mean-square pressure. The formulation derived by Cook *et al.* is for incoherent reverberant fields[9] (e.g., narrow-band noise excitation) and is valid for any fixed pair of points in space. The results can be utilized for the investigation of certain aspects of "diffusion" and to provide information about the acoustic inputs to structures being tested in reverberation chambers.

## II. COVARIANCE OF MEAN-SQUARE PRESSURE

As has been stated before, to ensure independent sampling of $\overline{p^2}$ it is necessary that the spatial covariance between paired samples be negligibly small. Hence the form of this covariance has to be determined. The mean-square pressure is usually defined as

$$\overline{p^2}(\mathbf{r}_1) = \lim_{T\to\infty} \frac{1}{T}\int_0^T p^2(\mathbf{r}_1, t)dt. \tag{10}$$

Therefore

$$\langle \overline{p^2}(\mathbf{r}_1)\,\overline{p^2}(\mathbf{r}_1)\rangle = \lim_{T\to\infty}\frac{1}{T^2}\int_0^T\int_0^T \langle p^2(\mathbf{r}_1, t_1)p^2(\mathbf{r}_2, t_2)\rangle dt_1\, dt_2. \tag{11}$$

As has been demonstrated in Sec. IB, the joint probability density of pressure is bivariate Gaussian. Thus the covariance of pressure squared can be written as (see, for example, Davenport and Root[14]),

$$\langle p^2(\mathbf{r}_1, t_1)p^2(\mathbf{r}_2, t_2)\rangle = \langle p^2(\mathbf{r}_1, t_1)\rangle\langle p^2(\mathbf{r}_2, t_2)\rangle + 2\langle p(\mathbf{r}_1, t_1)p(\mathbf{r}_2, t_2)\rangle^2. \tag{12}$$

Substituting Eqs. (12) and (11) in Eq. (4), it follows that

$$R(r) = 2\lim_{T\to\infty}\frac{1}{T^2}\int_0^T\int_0^T \langle p(\mathbf{r}_1, t_1)p(\mathbf{r}_2, t_2)\rangle^2 dt_1\, dt_2. \tag{13}$$

Using Eq. (9) for the space-time cross correlation of pressure, it is easy to show that

$$R(r) = (N/2)^2[\sin(kr)/kr]^2. \tag{14}$$

The normalized covariance is

$$\tilde{R}(r) = [\sin(kr)/kr]^2. \tag{15}$$

This result was first obtained by Lubman.[2]

## III. DISCUSSION

The normalized covariance of the mean-square pressure $\overline{p^2}$ is diminishing beyond $kr = \pi$, according to Eq. (15). Thus if $\overline{p^2}$ is measured at points half a wavelength or more apart, the samples can be considered to be uncorrelated. They cannot, however, be regarded as independent because the probability density of $\overline{p^2}$ is not Gaussian. They are exponentially distributed as indicated by Eq. (6).

The condition of independent sampling can be approached if some additional averaging schemes are incorporated. For example, averaging over many independent source positions (at least half a wavelength apart) is a possibility. As has been shown by Waterhouse,[4] the probability density function of the averaged $\overline{p^2}$ approaches normal if the number of positions is large. A similar situation results if averaging over modal pattern and frequency is applied by the use of a rotating diffuser. The case of narrow-band noise excitation can also be considered as averaging over frequency. It is necessary, however, to ensure that there are no significant changes to the form of the covariance function as a result of frequency averaging. It is shown in the Appendix that the form of the covariance function changes insignificantly when averaged over a bandwidth of 1/3 octave.

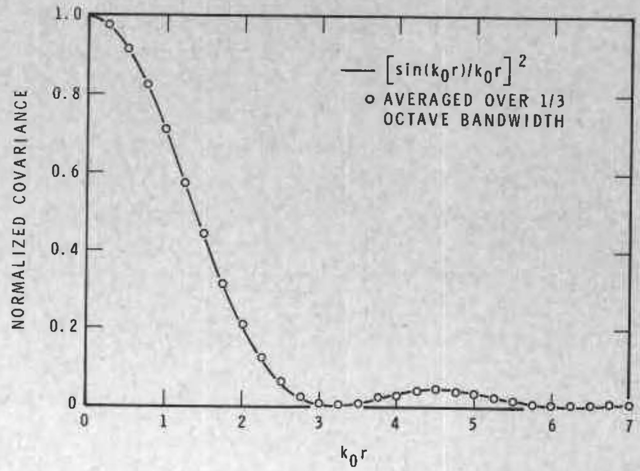


FIG. A1. Effect of bandwidth on normalized covariance function.

## APPENDIX: EFFECT OF FREQUENCY AVERAGING

Under narrow-band noise excitation the normalized covariance function of the mean-square pressure can be obtained by averaging Eq. (15) over frequency; i.e.,

$$\langle\tilde{R}(r)\rangle_f = (k_2 - k_1)^{-1}\int_{k_1}^{k_2}\left(\frac{\sin(kr)}{kr}\right)^2 dk. \tag{A1}$$

By expanding the integrand into a Taylor series about the mean wavenumber $[k_0 = (k_2 + k_1)/2]$ it can be shown that

$$\langle\tilde{R}(r)\rangle_f = F(k_0r) + \left(\frac{1}{24}\right)F''(k_0r)(k_0r)^2\left(\frac{k_2-k_1}{k_0}\right)^2 + \left(\frac{1}{1920}\right)F''''(k_0r)(k_0r)^4\left(\frac{k_2-k_1}{k_0}\right)^4 + \cdots, \tag{A2}$$

where $F(k_0r) = [\sin(k_0r)/k_0r]^2$ and $F''$ and $F''''$ are the second and fourth derivatives of $F$, respectively. $\langle\tilde{R}(r)\rangle_f$ has been computed for the case of $(k_2 - k_1)$ equal to 1/3 octave wide. Figure (A1) shows a comparison of $\langle\tilde{R}(r)\rangle_f$ with $[\sin(k_0r)/k_0r]^2$. The conclusion is that no significant changes have occurred as a result of frequency averaging.


[1]M. R. Schroeder, "Effect of frequency and space averaging on the transmission responses of multimode media," J. Acoust. Soc. Am. **46**, 277–283 (1969).

[2]D. Lubman, "Spatial averaging in a diffuse sound field," J. Acoust. Soc. Am. **46**, 532–534 (1969).

[3]M. R. Schroeder, "Spatial averaging in a diffuse sound field and the equivalent number of independent measurements," J. Acoust. Soc. Am. **46**,

534 (1969).
[4]R. Waterhouse, "Spatial properties of reverberant sound fields," J. Acoust. Soc. Am. **43**, 1436–1444 (1968).
[5]R. Waterhouse, "Spatial properties of reverberant sound fields," 6th Int. Congr. Acoust., Tokyo, Japan, Paper E-5-1 (1968).
[6]R. Waterhouse and D. Lubman, "Discrete versus continuous space averaging in a reverberant sound field," J. Acoust. Soc. Am. **48**, 1–5 (1970).
[7]D. Lubman, R. Waterhouse, and C. Chien, "Effectiveness of continuous spatial averaging in a diffuse sound field," J. Acoust. Soc. Am. **53**, 650–659 (1973).
[8]C. Chien, R. Waterhouse, and D. Lubman, "Spherical averaging in a diffuse sound field," J. Acoust. Soc. Am. **57**, 972–975 (1975).
[9]W. T. Chu, "Comments on the coherent and incoherent nature of a reverberant sound field," J. Acoust. Soc. Am. **69**, 1710–1715 (1981).
[10]M. R. Schroeder, "Frequency autocorrelation functions of frequency responses in rooms," J. Acoust. Soc. Am. **34**, 1819–1823 (1962).
[11]F. Jacobsen, "The diffuse sound field," Acoust. Lab., Tech. Univ. Denmark, Rep. No. **27** (1979).
[12]J. H. Laning, Jr., and R. H. Battin, *Random Processes in Automatic Control* (McGraw-Hill, New York, 1956).
[13]Reference 12, p. 85.
[14]W. B. Davenport, Jr., and W. L. Root, *An Introduction to the Theory of Random Signals and Noise* (McGraw-Hill, New York, 1958), p. 168.
[15]R. K. Cook, R. V. Waterhouse, R. D. Berendt, S. Edelman, and M. C. Thompson, Jr., "Measurement of correlation coefficients in reverberant sound fields," J. Acoust. Soc. Am. **27**, 1072–1077 (1955).