# Fundamentals of soil action under vehicles (part two)

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NATIONAL RESEARCH COUNCIL OF CANADA

ASSOCIATE COMMITTEE ON SOIL AND SNOW MECHANICS.

TECHNICAL MEMORANDUM NO. 8

# FUNDAMENTALS OF SOIL ACTION UNDER VEHICLES

(PART TWO)



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OTTAWA, CANADA.

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## ABSTRACT

This paper presents a mathematical analysis of the stability of a model representing a wheel or track of a moving vehicle.

The proposed method of determining a trafficability curve is based on accepted theories of Soil Mechanics and represents a continuation of the work described in Technical Memorandum No. 6.

A cohesionless medium and the stability of a single wheel or a single track shoe with or without grouser is alone considered. The proposed method, however, is general in scope, and may be easily extended over cohesive soils and several wheels and shoes.

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# INTRODUCTION

This work presents further development of the theory on the relationship between the Soil and Vehicle Models as presented in the Technical Memorandum No. 6 by R.F. LEFFET and M.G. BEKKER, published in October 1946 by the NATIONAL RESEARCH COUNCIL OF CANADA.

The analysis outlined in the present work is based on well established methods of Soil Mechanics. Students of this paper would be assisted in following the line of thought by reading "Theoretical Soil Mechanics" by Karl TERZAGHI (Chapman and Hall Ltd., London, and John Wiley and Son, New York - 1944) from which chapters Nos. VII and VIII are especially recommended: "Passive Earth Pressure" (Pages 100-105, 108-110) and "Bearing Capacity" (Pages 120-129).

Mathematical symbols used in this paper are those accepted by the American Society of Civil Engineers, and quoted by TERZAGHI (Pages XI-XII of his "Theoretical Soil Mechanics"). A few new symbols which result from the specific character of the work have been explained in the text.

# PROBLEM

It is known that the stability of a two-dimensional model, representing a wheel or track shoe with or without a grouser, is subject to laws of Soil Mechanics which determine the stability of a continuous footing.

It has been demonstrated already that the stability of such a model in a cohesionless soil depends on the angle of internal friction $\phi$ and the specific weight $\gamma$ of the soil. It also depends on the shoe length $s$ grouser depth $h$, vertical load $V$ and horizontal pull $H$ applied to the model.

It has been revealed also that the relationship between $H$ and $V$ cannot be expressed by a single function of H = F ($V \phi \gamma s h$) type, but that there are two distinct functions which are called TRAFFICABILITY FUNCTIONS. This led to new definitions of two different types of vehicle failure in negotiating the given terrain, namely to the definition of a GRIP and of a GROUND FAILURE.

A function which defines conditions for a grip failure has been determined previously in the form of the following equation:

$$H = V \frac{h + s\tan\phi}{s - h\tan\phi}$$

where the weight of soil "enclosed" by the shoe and grouser has been neglected, as it is insignificant.

The above equation enables one to calculate immediately the drawbar pull $H$ which causes the grip failure, once the vertical load $V$, friction angle $\phi$ and model dimensions $s$ and $h$ are given.

The second function which defines conditions of ground failure was determined graphically by applying the Logarithmic Spiral method. This function was traced in a few particular cases and resembled a hyperbola of an unspecified equation.

The graphic method adopted, as well as any other similar method applicable in this field requires considerable amount of work and involves a great deal of cumbersome computation. It appears advisable therefore, that a mathematical analysis of the stability of a model be made with the purpose of determining an approximate equation of ground failure.

This would not only shorten the work required for determining both types of TRAFFICABILITY CURVES, but also will form a broader basis for future discussion of the stability of vehicles and their performance.

## P R O P O S E D   S O L U T I O N

Take an indefinitely long grouser plate which is shown in Fig. 1. The action of this plate loaded with a vertical load V lbs. and a horizontal pull H lbs. may be replaced by the action of an imaginary continous footing $\overline{ab}$ which is sloped to the horizon at an angle.

$$\theta = \tan^{-1}\left(\frac{H}{V}\right)$$

In this case trajectories of principal stresses have their foci in points a and b. If the angle enclosed between the shoe plate, s and the hypotenuse $\overline{ga}$ is $\beta$ and if the angle of internal friction is $\phi$, then the angles, referred to the dimensions and to the location of the imaginary footing $\overline{ab}$, are those shown in Fig. 1.

As in the classical case which determines the bearing capacity of a strip load that has a width $\overline{ab}$, the task is to determine the passive earth resistance on retaining walls $\overline{da}$ and $\overline{db}$.

Since wall $\overline{db}$ is the weaker element of our structure, it may be assumed that its failure determines the stability limit of our model. In consequence the bearing capacity of the strip load $\overline{ab}$ is assumed to be equal

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| 5 | 10 | Equation should be numbered | 1 |
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| 8 | Last line | " " " " | 13 |

to **twice the** passive earth pressure exercised by the soil on the wall $\overline{db}$. This assumption involves certain errors which are however, on the safe side, and do not appear to distort the final result which may be checked by experiments.

Consider, first, a component $P_p'$ of the passive earth pressure due to the surcharge exercised by the soil layer which has the thickness $k_1$ (Fig. 1).

The magnitude of this component has been shown by Terzaghi. Respectively modified, it is:

$$P_p' = \frac{h_1}{\sin(\phi+\theta)\cos\phi}\, k_1 \gamma K_{p\gamma}$$

where $h_1$ is the depth of the retaining wall, and $K_{p\gamma}$ is a pure number whose value does not depend on $\gamma$ or on the depth $h_1$.

Respective values of $k_1$ and $h_1$ may be determined from Fig. 1

$$\overline{cb} = \frac{\sqrt{h^2+s^2}}{2}\cos(\beta-\theta)$$

hence

$$\overline{bd} = \frac{\sqrt{h^2+s^2}}{2\cos\phi}\cos(\beta-\theta)$$

and

$$h_1 = \overline{bf} = \frac{\sqrt{h^2+s^2}}{2\cos\phi}\cos(\beta-\theta)\sin(\phi+\theta)$$

but

$$\cos\beta = \frac{s}{\sqrt{h^2+s^2}}$$

$$\sin\beta = \frac{h}{\sqrt{h^2+s^2}}$$

And finally

$$h_1 = \frac{s}{2\cos\phi}\left(\cos\theta + \frac{h}{s}\sin\theta\right)\sin(\phi+\theta)$$

In a similar way:

$$\overline{bg} = \sqrt{h^2+s^2}\,\sin(\beta-\theta)$$

and

$$\overline{bj} = k_1 = \overline{bg}\cos\theta = \sqrt{h^2+s^2}\,\sin(\beta-\theta)\cos\theta$$

or, after having expressed $\sin\beta$, $\cos\beta$ by respective functions of $h$ and $s$

$$k_1 = s\cos\theta\left(\frac{h}{s}\cos\theta - \sin\theta\right)$$

The double component of $P_p'$ from equation 1 may be then determined by combining this equation with equations 2 and 3.

$$2P_p' = \frac{s^2\gamma\cos\theta}{\cos^2\phi}K_{p\theta}\left(\cos\theta + \frac{h}{s}\sin\theta\right)\left(\frac{h}{s}\cos\theta - \sin\theta\right)$$

If the model is buried at a constant depth $k_2$ (Fig. 2) which does not change with the angle $\theta$ as in the previous case, then the additional component of the passive earth pressure due to the above constant surcharge is (Terzaghi):

$$(P_p') = \frac{h_1}{\sin(\phi+\theta)\cos\phi}\, k_2\gamma K_{p\theta}$$

A doubled value of this component may be obtained easily by combining equations 5 and 2:

$$2(P_p') = \frac{s\,k_2\,\gamma}{\cos^2\phi}K_{p\theta}\left(\cos\theta + \frac{h}{s}\sin\theta\right)$$

The double value of passive earth resistance, which is due to the weight of soil and to its internal friction is (Terzaghi):

$$2P_p'' = \frac{h_1^2}{\sin(\phi+\theta)\cos\phi}\,\gamma K_{p\gamma}$$

Where $K_{p\gamma}$ is again a pure number whose value does not depend on specific weight of soil $\gamma$ and the depth at which the pressure is exercised.

By substituting $h_1$ from equation 2 in equation 7 the following result is obtained:

$$2P_p'' = \frac{\gamma s^2}{4\cos^3\phi}K_{p\gamma}\left(\cos\theta + \frac{h}{s}\sin\theta\right)^2\sin(\phi+\theta)$$

Finally, total passive earth pressure which determines the bearing capacity of our model is the sum of equation 4, 6 and 8.

$$2P_{p\theta} = 2(P_p') + 2P_p' + 2P_p''$$

or

$$2P_{p\theta} = \frac{K_{p\theta}\gamma s k_2\left(\cos\theta + \frac{h}{s}\sin\theta\right)}{\cos^2\phi} + \gamma s^2\left[\frac{K_{p\theta}\left(\cos\theta + \frac{h}{s}\sin\theta\right)\left(\frac{h}{s}\cos\theta - \sin\theta\right)\cos\theta}{\cos^2\phi} + \frac{K_{p\gamma}\left(\cos\theta + \frac{h}{s}\sin\theta\right)^2\sin(\phi+\theta)}{4\cos^3\phi}\right]$$

To simplify this expression let:

$$K_{P\delta}\left[\frac{\cos\theta + \frac{h}{s}\sin\theta}{\cos^2\phi}\right] = m_{sh\theta}$$

$$K_{P\delta}\left[\frac{(\cos\theta + \frac{h}{s}\sin\theta)(\frac{h}{s}\sin\theta - \cos\theta)\cos\theta}{\cos^3\phi}\right] + K_{P\gamma}\left[\frac{(\cos\theta + \frac{h}{s}\sin\theta)^2 \sin(\phi+\theta)}{4\cos^3\phi}\right] = n_{sh\theta}$$

Then, equation 9 may be written in the following form:

$$2P_{P\theta} = m_{sh\theta}\,\gamma s k_2 + n_{sh\theta}\,\gamma s^2 \qquad \text{.............10}$$

Values $m_{sh\theta}$ and $n_{sh\theta}$ depend on ratio $h/s$ angles $\phi$ and $\theta$. $K_{p\gamma}$ as well as $K_{P\delta}$ depend only on $\phi$ and $\theta$, therefore values of $m_{sh\theta}$ and $n_{sh\theta}$ may be determined graphically once and for all, for any given $h/s$, $\phi$ and $\theta$.

Values $m_{sh\theta}$ and $n_{sh\theta}$ were computed by the writer by means of a logarithmic spiral method for $\phi = 35^\circ$, various angles $\theta$ and for $h/s$ = 0, 0.25, 0.5, 0.75, 1. Respective figures are plotted in graph shown in Fig. 4 and 5.

The above graph and formula 10 enable one to trace the trafficability curve for any model of the above specified $h/s$ ratios in cohesionless sand, the friction of which is $\phi = 35^\circ$. This may be done in the following way:

From the condition of equilibrium of the model it follows that: (Fig. 1).

$$(2P_{P\theta})^2 = H^2 + V^2$$

and hence

$$H = (m_{sh\theta}\,\gamma s k_2 + n_{sh\theta}\,\gamma s^2)\sin\theta$$

$$V = (m_{sh\theta}\,\gamma s k_2 + n_{sh\theta}\,\gamma s^2)\cos\theta$$

It then becomes evident that the required function which defines the ground failure may be easily determined in polar coordinates, namely in an angle $\theta$ and in a radius $\rho = 2P_{P\theta}$

For this purpose it is sufficient to calculate a series of radii $2P_{P\theta}$ for various $\theta$ from equation 10 (Fig. 3). By connecting the ends of these radii the trafficability curve which defines the critical loads $H$ and $V$ at the moment of ground failure for given $h$, $s$ and $\phi$ may be readily obtained. Loads $H$ and $V$ will be determined from equations 11 and 12, or merely by projecting the respective vectors $2P_{P\theta}$ on $H$ and $V$ axis of Cartesian coordinates which have their zero point at the beginning of our polar coordinates (Fig. 3).

Attempts to eliminate parameter $\theta$ from equations 11 and 12 in order to obtain a single equation in Cartesian coordinates $H$ and $V$ seems to be futile as the form of this latter equation $H=F(V\phi\gamma s h)$ is too cumbersome for practical application.

Equation 10 may be used as a basis for general consideration. For instance it may be interesting to know when the Grip Failure ceases and when Ground Failure begins under given conditions. In other words the position of point A (Fig. 3) is to be determined.

In order to do this, it should be noticed that the required angle $\theta$ of radius $\overline{OA}$ may be determined from the formula.

$$\theta = \tan^{-1}\left(\frac{h + s\tan\phi}{s - h\tan\phi}\right)$$

The corresponding values $m_{sh\theta}$ and $n_{sh\theta}$ may be immediately ascertained from graphs shown in Fig. 4 and 5, and the required radius $2P_{P\theta} = \overline{OA}$ may be calculated easily from equation 10.

Another point of interest may be the maximum obtainable drawbar pull (H max.), with reference to the vertical load $V$, available for given conditions ($\phi$, $h$ and $s$). Since the differential $dH/dV$ cannot be obtained directly, equations 11 and 12 must be differentiated with reference to $\theta$. By dividing $dH/d\theta$ by $dV/d\theta$ and by assuming the result equal to zero a value $\theta$ may be found. To do this it is sufficient to solve the $\frac{dH}{d\theta} = 0$ equation:

$$\frac{dH}{d\theta} = \left(\frac{dm}{d\theta} k_2 + \frac{dn}{d\theta} s\right)\sin\theta + (mk_2 + ns)\cos\theta = 0$$

In order to find $\theta$ from this formula, functions $m = f(\theta)$ and $n = \varphi(\theta)$ must be known. As these functions have been determined graphically (Fig. 4 and 5) equation 13 may also be solved graphically by designating $\frac{dm}{d\theta}$ and $\frac{dn}{d\theta}$ values as tangents of angles which are enclosed between $\theta$ axis and respective lines tangent to $m$ and $n$ curves as shown in Fig. 4 and 5.

The graph plotted in Fig. 6 contains equation 13 solved for $k_2 = s$. The results indicate that for a surcharge $k_2$ equal to the width of the shoe, the maximum drawbar pull $H$ may be obtained if the angle $\theta$ which is determined by the corresponding $H/V$ ratio takes the following values:

| | | |
|---|---|---|
| For a plate: | $h/s = 0$ | $\theta = 23^\circ$ |
| For grouser plates: | $h/s = .25$ | $\theta = 23^\circ\ 45'$ |
| | $h/s = .5$ | $\theta = 24^\circ\ 30'$ |
| | $h/s = .75$ | $\theta = 26^\circ\ 0'$ |
| | $h/s = 1.0$ | $\theta = 27^\circ\ 20'$ |

If there is no surcharge $\frac{dm}{d\theta} = 0$ and $m = 0$. Equation 13 is much simplified and may be solved as shown in Fig. 7.

In this case the maximum obtainable drawbar pull takes place at the following $\theta$ values:

| | | |
|---|---|---|
| For a plate: | $h/s = 0$ | $\theta = 18^\circ\ 30'$ |
| For grouser plates: | $h/s = .25$ | $\theta = 19^\circ\ 30'$ |
| | $h/s = .5$ | $\theta = 20^\circ\ 30'$ |
| | $h/s = .75$ | $\theta = 22^\circ\ 20'$ |
| | $h/s = 1.0$ | $\theta = 24^\circ\ 15'$ |

For any of the above quoted $\theta$ angles the respective $m_{she}$ and $n_{she}$ may be found from graphs shown in Fig. 4 and 5.

By substituting these values in equations 11 and 12 the maximum drawbar pull (H max.) and the corresponding $V$ load may easily be obtained. Hence the location of the maximum point M (Fig.3) may be plotted.

A third matter of interest may be the point B (Fig. 3) in which soil fails due to the vertical load only. This point may be found by substituting in equation 11 and 12 $m_{she}$ and $n_{she}$ values for $\theta = 0$ (Fig. 4 and 5).

It will be noted that in this case formulae 10 and 12 reduce themselves to an equation which is identical to the equation given by TERZAGHI for the determination of the bearing capacity of a strip load. Coefficients $m_{sh\theta}$ and $n_{sh\theta}$ become identical with the TERZAGHI's values $N_q$ and $N_\gamma$ as quoted in his "Theoretical Soil Mechanics".

## NUMERICAL EXAMPLE

Determine polar coordinates and values of $H$ and $V$ in points A, M and B of the Trafficability curve (Fig. 3) for the following grouser plate: s = 5", h = 2.5". The plate whose width is w = 20", is acting without a surcharge upon a homogeneous cohesionless sand (friction $\phi$ = 35° and specific weight $\gamma$ = 0.06 lbs/cu. in.).

**Solution**

(a) Coordinates and loads of point A:

$$\theta = \tan^{-1}\left(\frac{2.5 + 5\tan 35^\circ}{5 - 2.5\tan 35^\circ}\right) \cong 61^\circ$$

The respective value of $n_{sh\theta}$ (for $h/s = 2.5/5 = .5$) is about 3 (Fig. 4). Hence the required radius $\overline{OA}$ is:

$$2P_{p\theta} = \overline{OA} = 3 \times 006 \times 5^2 \cong 4.5 \text{ lb/in}$$

and the polar co-ordinates c point A are: $\theta$ = 61°, $\rho$ = 4.5 lbs./in.

Corresponding drawbar pull H and the vertical load V for the total width w = 20" of the plate are:

$$H = 4.5 \times 20 \times \sin 61^\circ \cong 78.3 \text{ lbs}$$

$$V = 4.5 \times 20 \times \cos 61^\circ \cong 43.5 \text{ lbs}$$

(b) Co-ordinates of point M and maximum obtainable drawbar pull:

From data quoted in this paper, $h/s$ = 0.5 and $\phi$ = 35° give the maximum drawbar pull at $\theta$ = 20° 30' (no surcharge). The corresponding $n_{sh\theta}$ value is about 20 (Fig. 4). Hence $\rho$ max. is:

$$\rho_{max} = 20 \times .06 \times 5^2 = 30.0 \text{ lb/in}$$

Loads which can be safely supported are:

$$H_{max} = 30 \times 20 \sin(20^\circ 30') \cong 210 \text{ lbs.}$$

$$V = 30 \times 20 \cos(20^\circ 30') \cong 562 \text{ lbs.}$$

(c) Co-ordinates of point B and the vertical load V:

From the definition of point B, $\theta = 0^{o}$, and the corresponding $n_{sh\theta}$ value is about 44 (Fig. 4) hence:

$$S = 44 \times .06 \times 5^{2} \cong 66 \text{ lb/in}$$

and

$$V = 66 \times 20 \times \cos 0^{\circ} \cong 1320 \text{ lbs.}$$

## CONCLUSIONS

The above analytic method enables one to find all or three basic points (A,M,B - Fig. 3) of the Trafficability curve which determines the Ground Failure and thus makes it easy to trace this curve for any model.

In order to apply this method to all cohesionless soils, values $m_{sh\theta}$ and $n_{sh\theta}$ should be computed in the same way as was done in this paper for $\phi = 35^{o}$. The deduced formula appears to have the same meaning and limits of application as other similar formulae in Soil Mechanics determining the bearing capacity of soil.

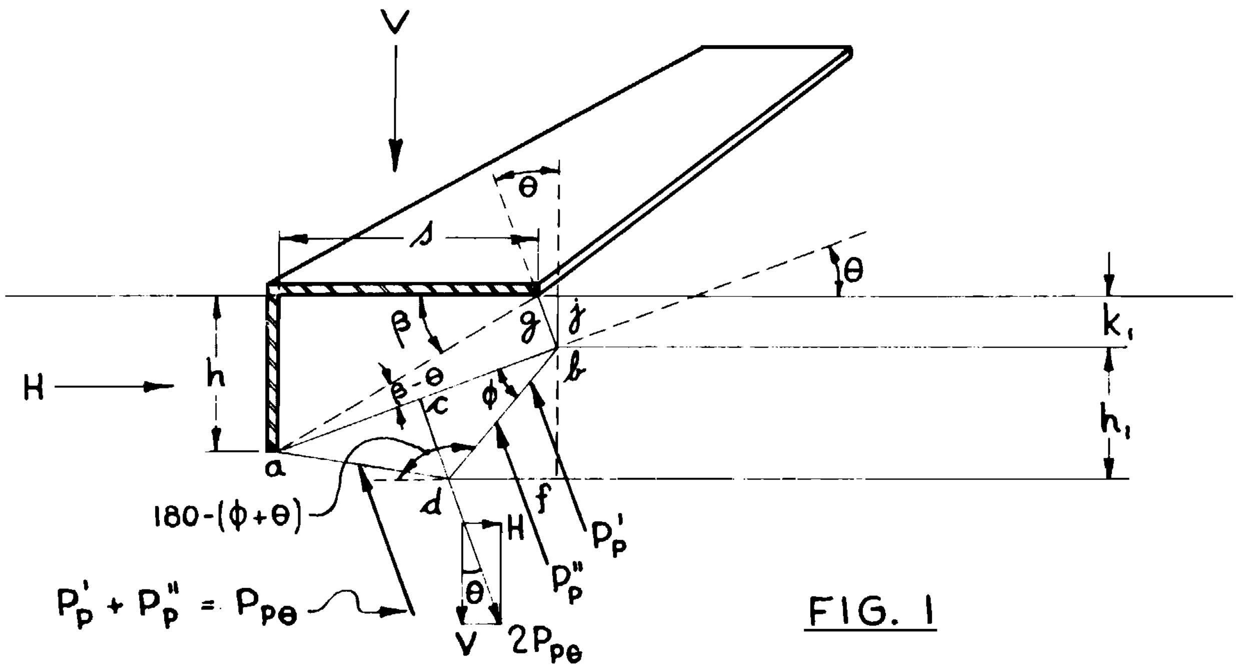



FIG. 1

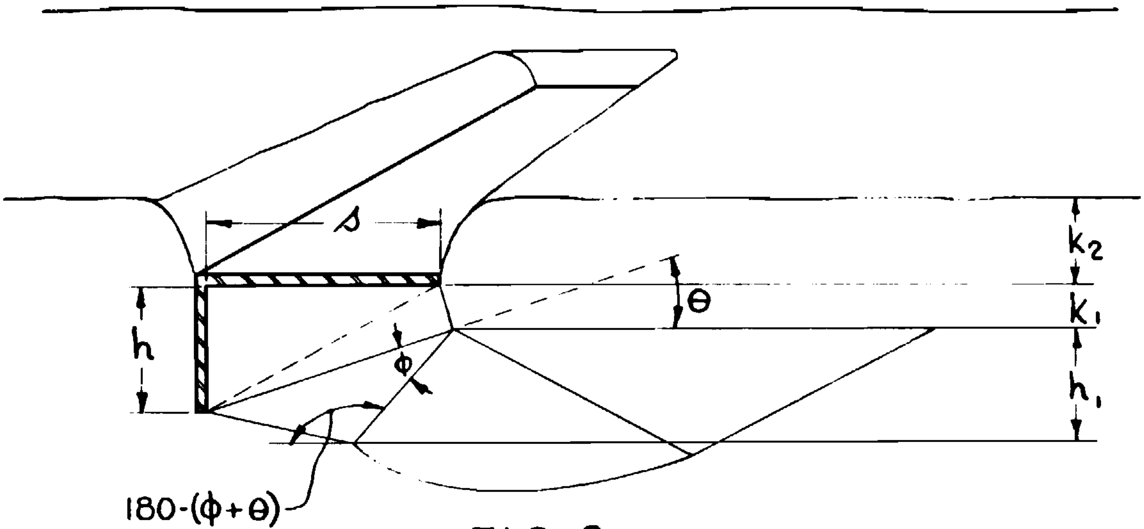



FIG. 2

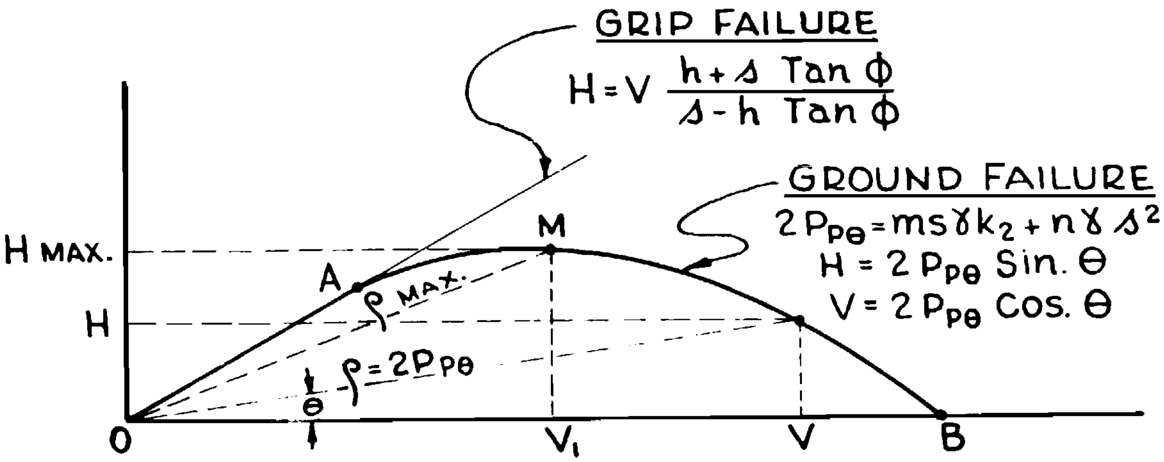



FIG. 3

T.P.S.

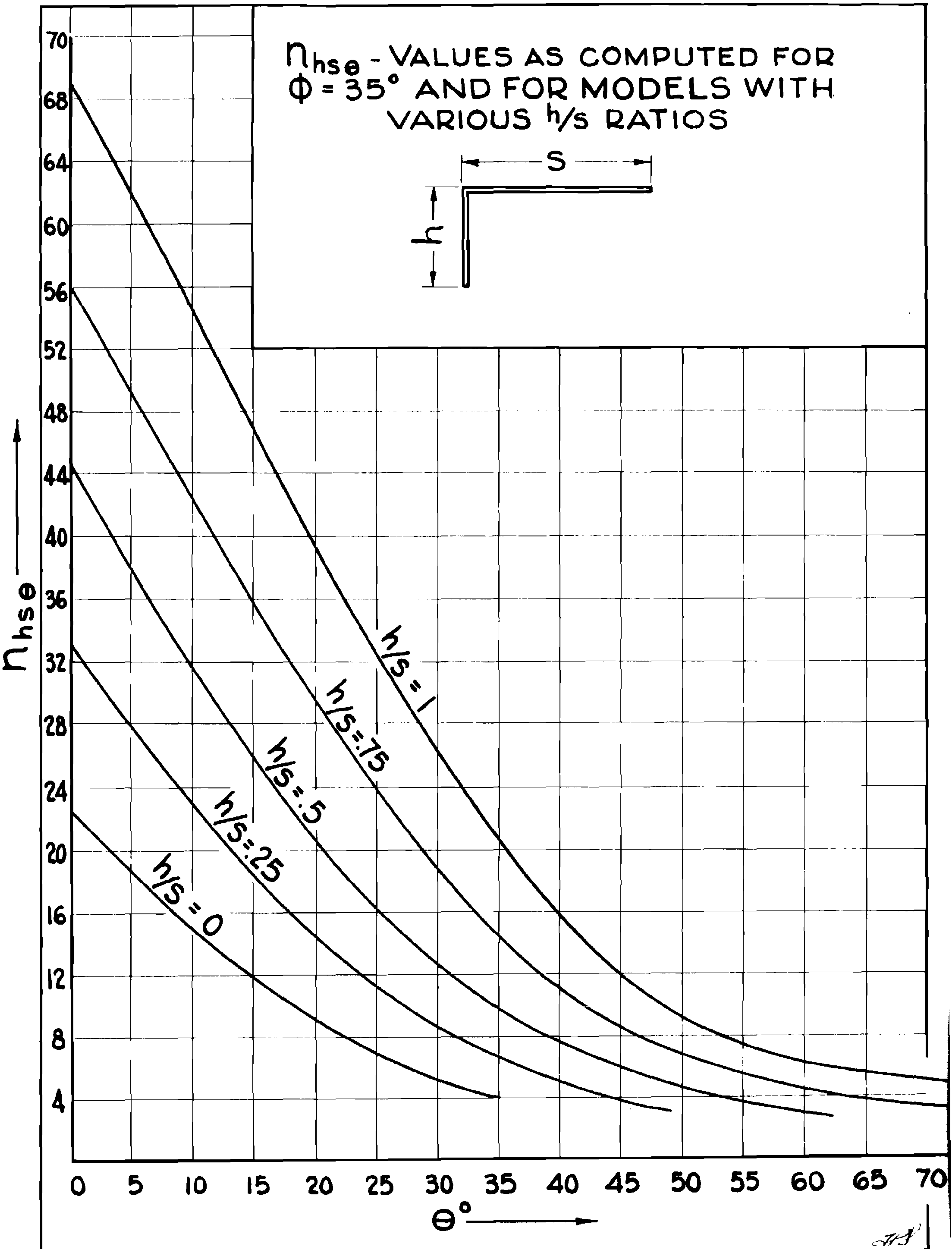
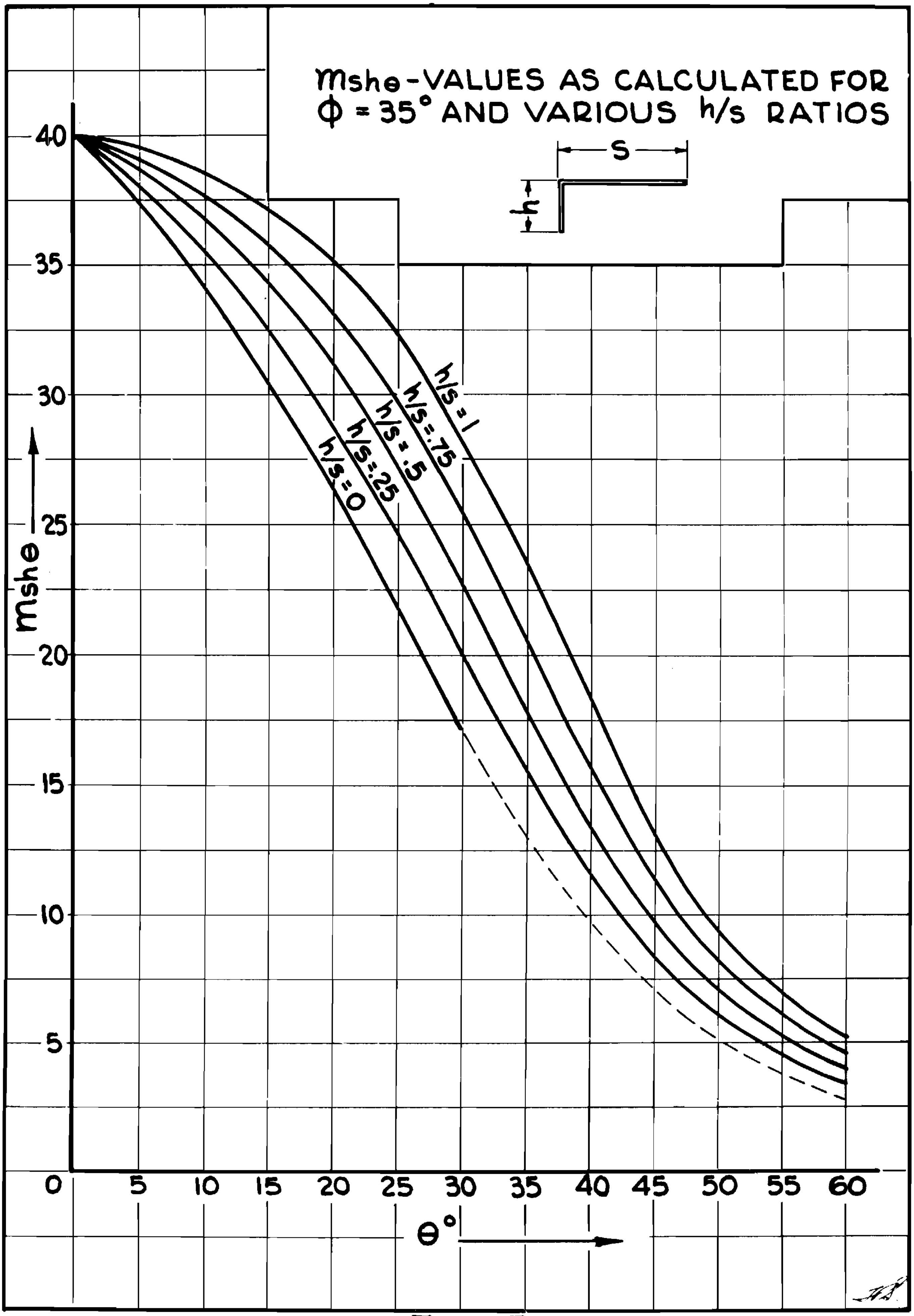
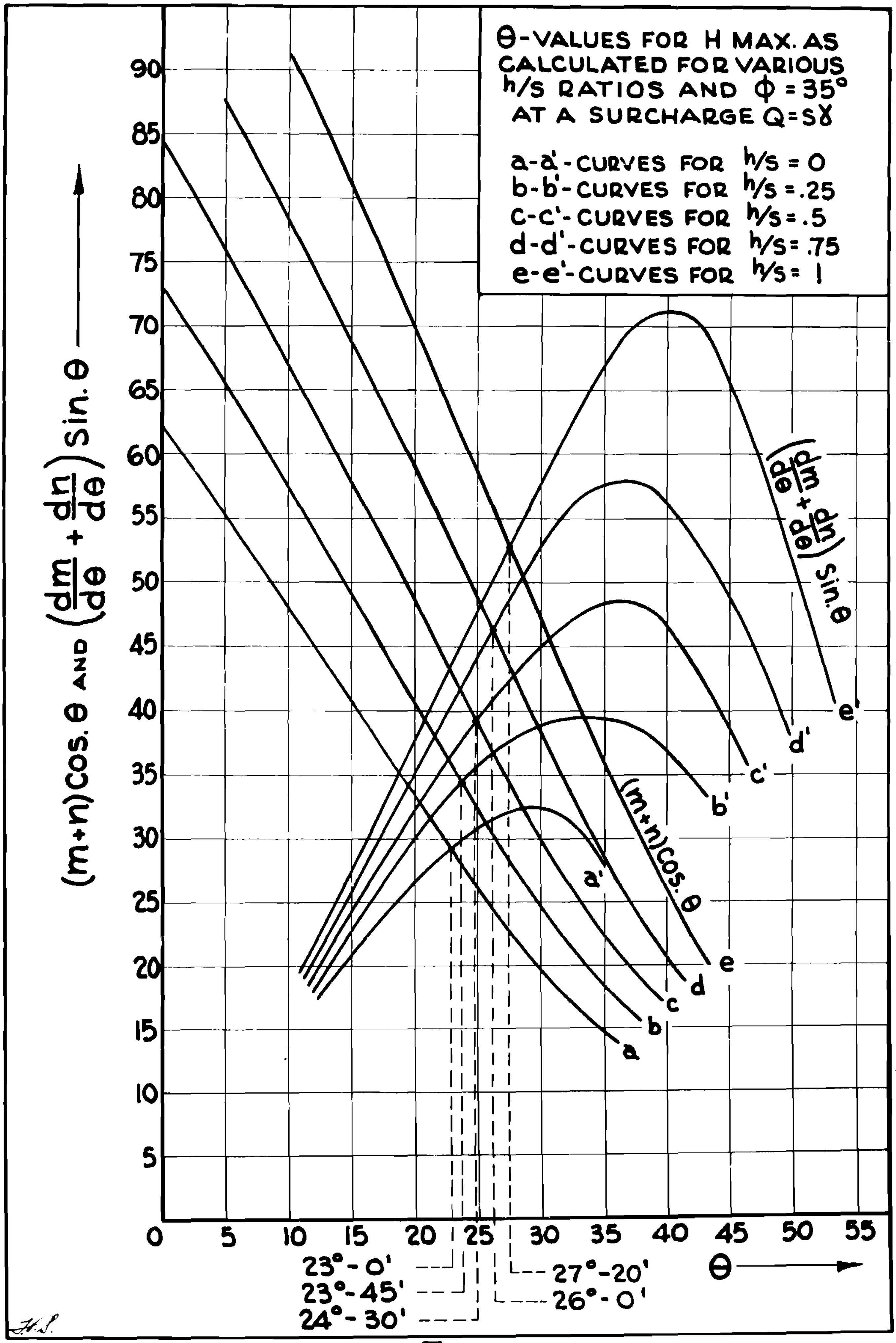

$n_{hs\theta}$ - VALUES AS COMPUTED FOR
$\phi = 35°$ AND FOR MODELS WITH
VARIOUS h/s RATIOS
S
h
$n_{hs\theta}$
h/s = 1
h/s=.75
h/s=.5
h/s=.25
h/s = 0
70
68
64
60
56
52
48
44
40
36
32
28
24
20
16
12
8
4
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70
$\theta°$

FIG. 4



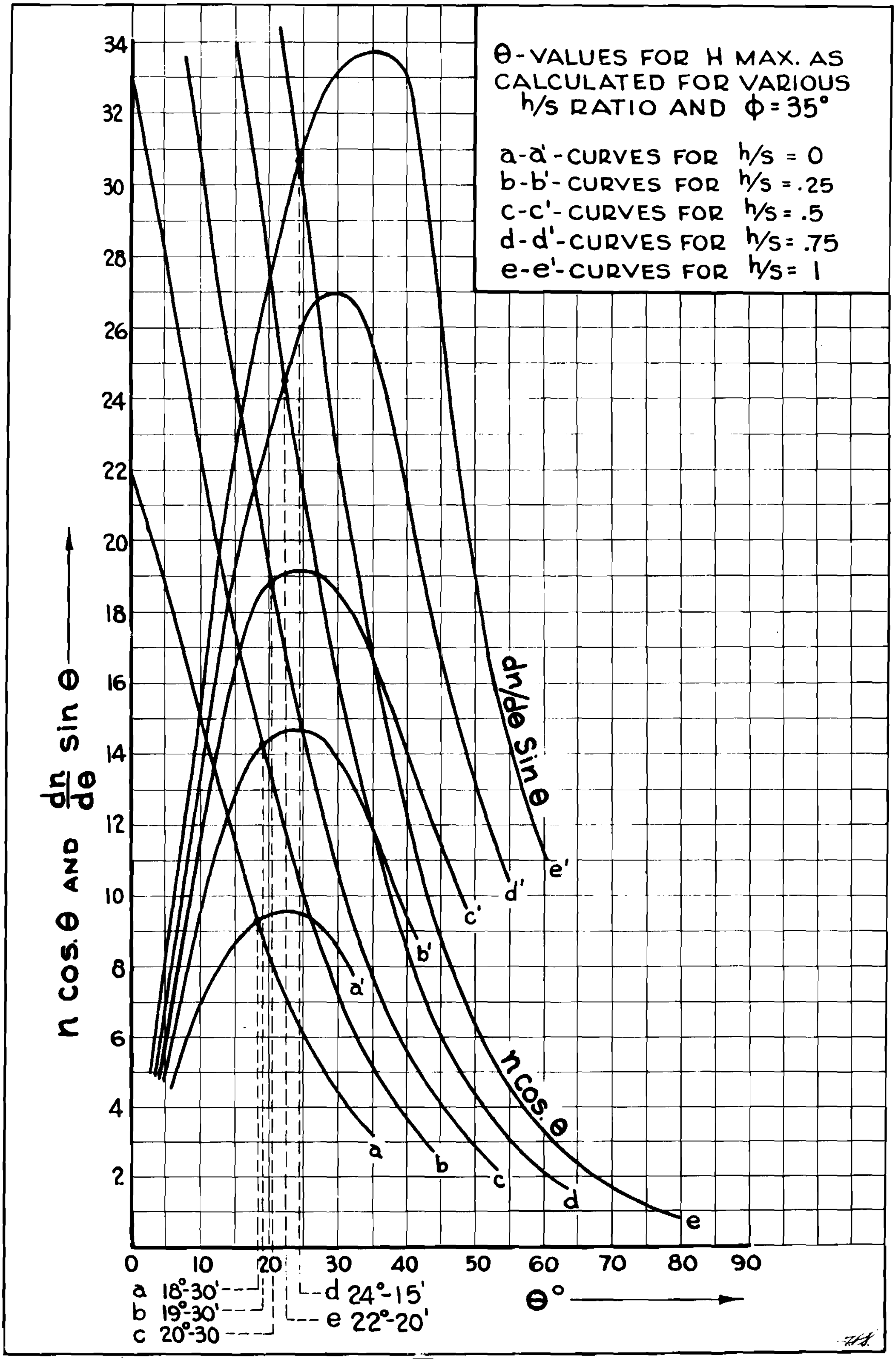
mshe-VALUES AS CALCULATED FOR
Φ = 35° AND VARIOUS h/s RATIOS
s
h
h/s = 1
h/s = .75
h/s = .5
h/s = .25
h/s = 0
mshe
θ°
0
5
10
15
20
25
30
35
40
45
50
55
60


FIG. 5



θ-VALUES FOR H MAX. AS CALCULATED FOR VARIOUS h/s RATIOS AND φ = 35° AT A SURCHARGE Q=Sγ
a-a'-CURVES FOR h/s = 0
b-b'-CURVES FOR h/s = .25
c-c'-CURVES FOR h/s = .5
d-d'-CURVES FOR h/s = .75
e-e'-CURVES FOR h/s = 1
(m+n) COS. θ AND (dm/dθ + dn/dθ) Sin. θ
(dm/dθ + dn/dθ) Sin. θ
(m+n) COS. θ
θ
23°-0'
23°-45'
24°-30'
26°-0'
27°-20'


FIG. 6



θ-VALUES FOR H MAX. AS CALCULATED FOR VARIOUS h/s RATIO AND φ=35°
a-a'-CURVES FOR h/s = 0
b-b'-CURVES FOR h/s = .25
c-c'-CURVES FOR h/s = .5
d-d'-CURVES FOR h/s = .75
e-e'-CURVES FOR h/s = 1
n cos.θ AND dn/dθ sin θ
dn/dθ sin θ
n cos.θ
θ°
a 18°-30'
b 19°-30'
c 20°-30
d 24°-15'
e 22°-20'


FIG. 7