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BY

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*Lumping Errors of Analog Circuits for Heat Flow Through a Homogeneous Slab**

ANALYZED

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ABSTRACT

This paper compares the transfer functions for several electrical analog circuits, both active and passive, with the theoretical functions for heat conduction through a homogeneous slab. The results indicate the magnitude of the errors due to space lumping as a function of non-dimensional frequency for each circuit. This permits the selection of the simplest circuit which will achieve a specified accuracy over a specified frequency range.

1. INTRODUCTION

An accurate calculation of the heat flux and temperature distribution through the walls, ceiling or floor of a room requires the simultaneous calculation of the room side surface heat flux of all the elements enclosing the room. Reference [1] discusses the use of an analog computer for this type of building heat transfer problem. It is pointed out in that paper that one of the important considerations in setting up a computer for a room heat transfer calculation is: "How to simulate accurately the wall, roof and floor sections with as few computer elements as possible." This paper presents a method of designing analog circuits to calculate one-dimensional heat conduction through a homogeneous slab with a specified accuracy. The analysis differs from earlier studies [2, 3, 4, 5] in two respects: it compares the frequency response of the analog circuits with the theoretical frequency response of a homogeneous slab; and it includes electronic analog circuits with their possibilities for using higher accuracy difference expressions, as well as the passive resistance-capacitance ladder networks.

Previous studies [2, 3, 4, 5] have considered the transient response of passive analog circuits and have concluded that many elements are required if an analog is to represent accurately the response of a slab to a sudden change in the driving function. Only the low frequency components of the outside surface temperature, however, have any significant effect on the conditions inside buildings. Thus any analog that is accurate for frequencies up to the third or fourth harmonic of a diurnal driving function is quite satisfactory for calculating conditions inside a building.

Designing an analog circuit for a limited frequency range leads to simpler circuits than are needed for an accurate response to a step change in the driving function. Thus the frequency response approach is used in this investigation.

Carslaw and Jaeger [6] show that when the temperatures at each surface of a homogeneous slab vary sinusoidally, the surface temperatures and heat flows are related by linear equations which can be expressed as:

$$\begin{bmatrix} \theta_{out} \\ q_{out} \end{bmatrix} = \begin{bmatrix} A\ , & -RB \\ -\dfrac{D}{R}\ , & A \end{bmatrix} \cdot \begin{bmatrix} \theta_{in} \\ q_{in} \end{bmatrix} \qquad (1)$$

*Tables I–IV have been deposited as Document 6679 with the American Documentation Institute Publications Project. A copy may be secured by citing the Document number and by remitting $1.25 for photoprints, or $1.25 for 35 mm microfilm.

where θ = temperature

q = heat flux

$A = \text{Cosh}\,(1+j)\,\phi$

$B = \dfrac{\text{Sinh}\,(1+j)\,\phi}{(1+j)\,\phi}$

$D = (1+j)\,\phi\,\text{Sinh}\,(1+j)\,\phi$

$\phi = \sqrt{\dfrac{\pi L^2}{\alpha P}}$

$R = L/k$

L = thickness of the slab

k = thermal conductivity

α = thermal diffusivity

P = period of the temperature cycle

The square matrix on the right side of (1) is called the transmission matrix for the slab.

Equation (1) can be rearranged to give:

$$R \cdot \begin{bmatrix} q_{\text{in}} \\ q_{\text{out}} \end{bmatrix} = \begin{bmatrix} A/B\ , & -1/B \\ 1/B\ , & -A/B \end{bmatrix} \cdot \begin{bmatrix} \theta_{\text{in}} \\ \theta_{\text{out}} \end{bmatrix} \tag{2}$$

The D term has been eliminated by using the fact that the determinant

$$\begin{vmatrix} A & B \\ D & A \end{vmatrix} = 1 \tag{3}$$

If an analog circuit is to calculate accurately the heat flux at the surfaces of a slab it must have transfer functions which are similar to A and B for all frequencies up to the frequency of the highest non-negligible harmonic of the driving temperatures. In the following sections, the A and B transfer functions are presented for several different analog circuits.

2. ANALOG CIRCUITS TO CALCULATE TEMPERATURE AND HEAT FLOW THROUGH A SLAB

The temperature distribution through a homogeneous slab (Fig. 1) is described by

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}$$

where

x = space coordinate in the direction of heat flow

t = time

This partial differential equation can be solved approximately by an analog computer for the temperature at the planes $x = x_1, x_2 \text{----} x_{N-1}$. The method is to replace $\left(\dfrac{\partial^2\theta}{\partial x^2}\right)_{x = x_i}$ by a finite difference expression involving the temperatures at the planes $x = x_0, x_1 \text{----} x_N$. x_0 and x_N are the two boundaries and the temperatures at these planes are the necessary boundary conditions. The use of a finite difference expression for $\partial^2\theta/\partial x^2$ converts the partial differential equation into a set of ordinary differential equations which have to be solved simultaneously.

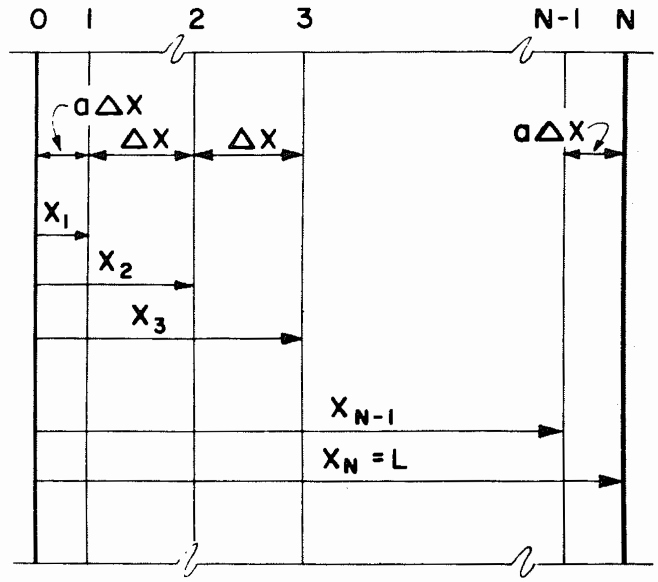

FIG. 1 SUBDIVISION OF A SLAB INTO N layers.

The heat flux through any plane $x = x_i$ is given by

$$q_i = -k\left(\frac{\partial\theta}{\partial x}\right)_{x = x_i} \tag{4}$$

Thus to evaluate heat flux with an analog computer, it is also necessary to approximate $\dfrac{\partial\theta}{\partial x}$ by a finite difference expression.

If one neglects the effects of imperfections in the computer components, analog circuits differ in their accuracy only because of the differences in the approximation of the space derivatives on which they are based. In Appendix I the following finite difference expressions for the space derivatives are derived:

$$-\Delta x\left(\frac{\partial\theta}{\partial x}\right)_0 = \frac{1+2a}{a(1+a)}\,\theta_0 - \frac{1+a}{a}\,\theta_1 + \frac{a}{1+a}\,\theta_2 - \frac{a(1+a)}{6}\,(\Delta x)^3\left(\frac{\partial^3\theta}{\partial x^3}\right)_1 - \frac{a(1+a)\,(1-2a)}{24}\times (\Delta x)^4\left(\frac{\partial^4\theta}{\partial x^4}\right)_1 \tag{5}$$

$$(\Delta x)^2\left(\frac{\partial^2\theta}{\partial x^2}\right)_1 = \frac{2}{a(1+a)}\,\theta_0 - \frac{2}{a}\,\theta_1 + \frac{2}{1+a}\,\theta_2 - \frac{(1-a)}{3}\times (\Delta x)^3\left(\frac{\partial^3\theta}{\partial x^3}\right)_1 - \frac{(1+a^3)}{12(1+a)}\,(\Delta x)^4\left(\frac{\partial^4\theta}{\partial x^4}\right)_1 \tag{6}$$

$$(\Delta x)^2\left(\frac{\partial^2\theta}{\partial x^2}\right)_i = \theta_{i-1} - 2\theta_i + \theta_{i+1} - \frac{(\Delta x)^4}{12}\left(\frac{\partial^4\theta}{\partial x^4}\right)_i \quad (7)$$

for $1 < i < N-1$

Equations (5) and (6) also apply at points N and $N-1$ respectively when Δx is replaced by $-\Delta x$ and the subscripts

0 by N,
1 by $N-1$, and
2 by $N-2$.

Δx is the distance between the equally spaced internal points and

$$a = \frac{x_1 - x_0}{\Delta x} = \frac{x_N - x_{N-1}}{\Delta x}.$$

The terms including the third and all higher derivatives are usually neglected. This constitutes the error in the finite difference approximations.

This error can be reduced by choosing the value of a so that the coefficients of the neglected derivatives are as small as possible. For instance, when $a = 1$, the coefficient of $\frac{\partial^3\theta}{\partial x^3}$ in (6) is zero so the first neglected term in the expression for $\frac{\partial^2\theta}{\partial x^2}$ is $\frac{(\Delta x)^2}{12}\left(\frac{\partial^4\theta}{\partial x^4}\right)$. This is the same for all the internal points. The error can be reduced by decreasing Δx, i.e. increasing the number of lumps. The first two terms which are neglected in the expression for the temperature gradient at the surface, however, are $\frac{(\Delta x)^2}{3}\left(\frac{\partial^3\theta}{\partial x^3}\right)_1 - \frac{(\Delta x)^3}{12}\left(\frac{\partial^4\theta}{\partial x^4}\right)_1$.

When $a = 0.5$, the third derivative term is reduced to $\frac{(\Delta x)^2}{8}\left(\frac{\partial^3\theta}{\partial x^3}\right)_1$ and the fourth derivative term dis-

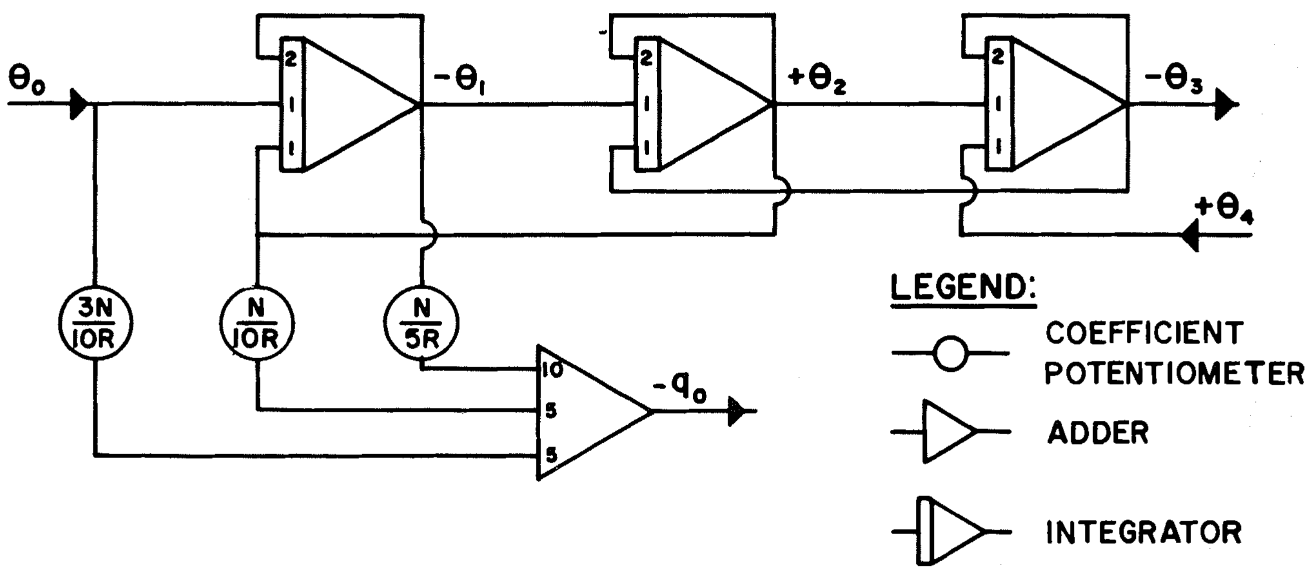

FIG. 2a ANALOG CIRCUIT TO COMPUTE SURFACE HEAT FLUX AND INTERNAL TEMPERATURES FOR A HOMOGENEOUS SLAB WHEN SURFACE TEMPERATURE IS GIVEN.

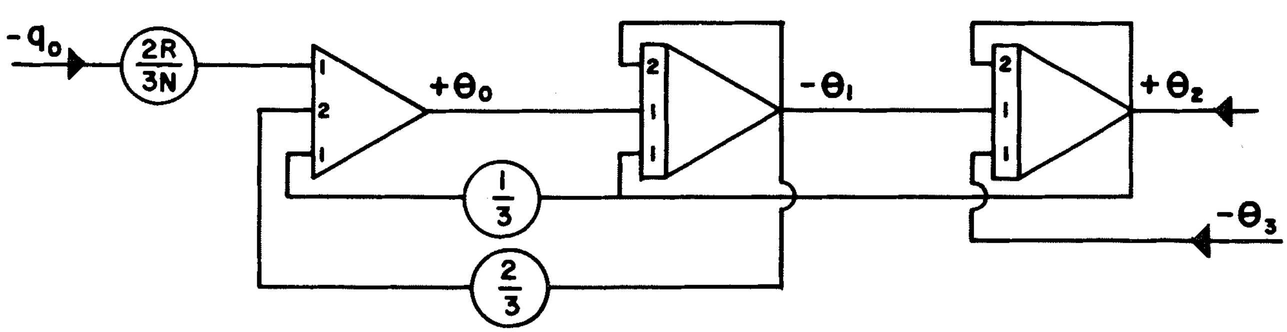

FIG. 2b ANALOG CIRCUIT TO COMPUTE SURFACE TEMPERATURE AND TEMPERATURES THROUGH A HOMOGENEOUS SLAB WHEN SURFACE HEAT FLUX IS GIVEN.

appears. This improvement in the accuracy of the expression for the surface gradient is partially offset by the reappearance of a $\frac{\Delta x}{6}\left(\frac{\partial^3\theta}{\partial x^3}\right)_1$ in the expression for $\left(\frac{\partial^2\theta}{\partial x^2}\right)_1$

The circuits for an operational amplifier type of analog shown in Figs. 2a and 2b are based on the difference equations for $a = 1.0$

$$\text{i.e.}\quad -\left(\frac{\partial\theta}{\partial x}\right)_0 = \frac{3\theta_0 - 4\theta_1 + \theta_2}{2\,\Delta x} \tag{8}$$

$$\text{and}\left(\frac{\partial^2\theta}{\partial x^2}\right)_i = \frac{\theta_{i+1} - 2\theta_i + \theta_{i+1}}{(\Delta x)^2} \tag{9}$$

for $1 \leq i \leq N-1$

$\Delta x = L/N$

The circuits for $a = 0.5$ are only slightly more complicated.

The values of the transmission matrix coefficients for this type of analog circuit have been calculated and the results for $a = 1.0$ are given in Table I, and for $a = 0.5$ in Table II. The expressions used for calculating these coefficients are derived in Appendix II. It is possible to obtain the matrix coefficients using an analog computer but the results obtained in this way include the errors due to imperfections of the computer components. The results calculated from the expressions in Appendix II represent the performance of an analog with perfect components. The significance of the differences between the matrix coefficients for an analog circuit and the theoretical coefficients for a homogeneous slab are discussed later.

A passive network of resistances and capacitances can also be used to simulate the heat flow in a slab. In this case the temperatures are represented by voltages and the heat flows by currents. Figure 3 is a typical passive analog circuit. The voltage at any internal point i is described by

$$\frac{d}{dt}(V'_i) = \frac{1}{R'C'}\left\{V'_{i+1} - 2V'_i + V'_{i+1}\right\} \tag{10}$$

Thus if $\frac{1}{R'C'} \propto \frac{\alpha}{(\Delta x)^2}$

this type of circuit will solve the set of ordinary differential equations for the temperatures through a slab.

With this type of analog the surface heat flux is represented by current; thus

$$q_0 \propto I_0 = \frac{V'_0 - V'_1}{aR'} \tag{11}$$

This is equivalent to using

$$-\left(\frac{\partial\theta}{\partial x}\right)_0 = \frac{\theta_0 - \theta_1}{a\cdot\Delta x} \tag{12}$$

A Taylor's series expansion about $x = 0$ gives

$$-\left(\frac{\partial\theta}{\partial x}\right)_0 = \frac{\theta_0 - \theta_1}{a\cdot\Delta x} + \frac{a(\Delta x)}{2!}\left(\frac{\partial^2\theta}{\partial x^2}\right)_0 + \dots \tag{13}$$

Thus taking I_0 as proportional to q_0 involves neglecting $\frac{a\cdot\Delta x}{2}\left(\frac{\partial^2\theta}{\partial x^2}\right)_0$ and all the higher derivatives. This higher error in the surface heat flux is an important disadvantage of the passive networks compared with the operational amplifier networks.

The A_N and B_N transmission matrix coefficients have been evaluated for resistance-capacitance networks with $a = 0.5$ for $N = 3$ (1) 14. (When $a = 0.5$ an N lump circuit consists of $N-1$ "T" networks in series.) The results are given in Table III. The

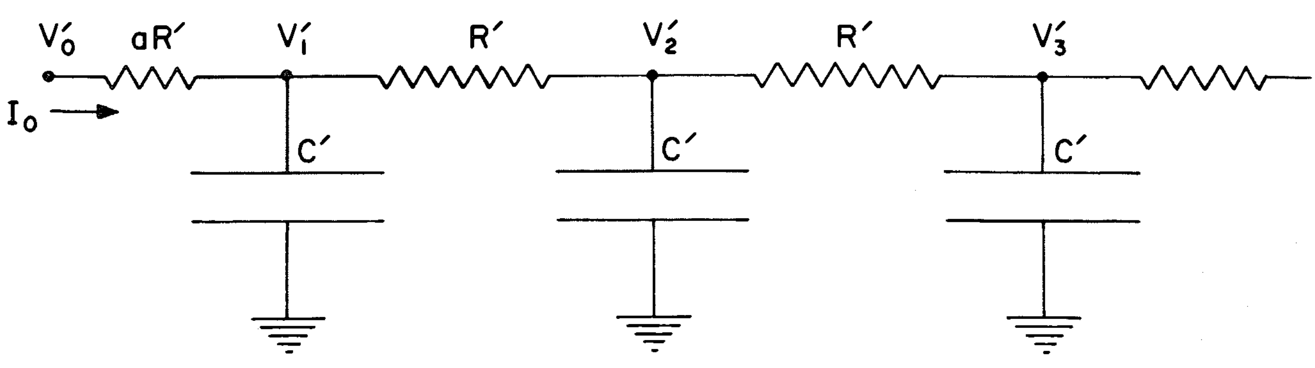

FIG. 3 RESISTANCE-CAPACITANCE CIRCUIT TO SIMULATE HEAT FLOW IN A HOMOGENEOUS SLAB.

method used was similar to the one outlined in Appendix II for the amplifier networks except that the expression for the temperature gradient at the surface was

$$\left(\frac{\partial \theta}{\partial x}\right)_N = \frac{\theta_N - \theta_{N-1}}{a \cdot \Delta x}$$

$$\text{where } \Delta x = \frac{L}{(N+2a-2)}$$

3. DISCUSSION OF RESULTS

The heat flows indicated by an analog circuit are given by

$$\begin{bmatrix} q_O \\ \\ q_N \end{bmatrix} = \frac{1}{R}\begin{bmatrix} \frac{A_N}{B_N} & , & \frac{-1}{B_N} \\ \\ \frac{1}{B_N} & , & -\frac{A_N}{B_N} \end{bmatrix} \cdot \begin{bmatrix} \theta_O \\ \\ \theta_N \end{bmatrix} \tag{14}$$

(The subscripts indicate that the quantity is associated with a lumped analog circuit.) If it is assumed that $\theta_O = \theta_{in}$ and $\theta_N = \theta_{out}$, equation (14) can be subtracted from (2) to give

$$\begin{bmatrix} q_{in} - q_O \\ \\ q_{out} - q_N \end{bmatrix} = \frac{1}{R}\begin{bmatrix} \frac{A}{B} - \frac{A_N}{B_N} , \frac{1}{B_N} - \frac{1}{B} \\ \\ \frac{1}{B} - \frac{1}{B_N} , \frac{A_N}{B_N} - \frac{A}{B} \end{bmatrix} \cdot \begin{bmatrix} \theta_{in} \\ \\ \theta_{out} \end{bmatrix} \tag{15}$$

The lefthand column matrix elements are the errors in the heat flux as calculated by the analog circuit.

$$\text{Let } \frac{A}{B} - \frac{A_N}{B_N} = U_N e^{j\delta_1} \tag{16}$$

$$\frac{1}{B} - \frac{1}{B_N} = V_N e^{j\delta_2} \tag{17}$$

$$\theta_{in} = |\theta_{in}|\, e^{jwt} \tag{18}$$

$$\theta_{out} = |\theta_{out}|\, e^{j(wt+\delta_0)} \tag{19}$$

The error in the analog heat flow will have its maximum possible value of

$$|q_{in} - q_0|_{max} = \frac{U_N\,|\theta_{in}| + V_N\,|\theta_{out}|}{R} \tag{20}$$

when $\delta_0 = \delta_1 - \delta_2 + \pi$

The error in the output heat flow is obtained by interchanging U_N and V_N in (20). Thus, the maximum possible error in the boundary heat flows indicated by the analog can be calculated easily when U_N and V_N are known.

The results of the matrix coefficient calculations for the operational amplifier and the resistance-capacitance circuits are plotted in Figs. 4a, 4b, and 5 in the form $\frac{A_N}{B_N}$, $\frac{1}{A_N}$ and $\frac{1}{B_N}$. The theoretical values for a homogeneous slab taken from reference 7 are also plotted so the values of U_N and V_N can be measured directly from the graphs. Values of U_N and V_N are given in Table IV as functions of ϕ and N for the various circuits. These data were measured off large scale graphs similar to Figs. 4 and 5.

The results in Table IV show that the operational amplifier type analog with $a = 0.5$ is the most accurate, of the circuits studied, for the computation of surface heat flows.

The following example illustrates how the charts and tables can be used to calculate the number of lumps needed for a particular problem.

Example 1.

Problem: How many lumps are required in an operational amplifier analog circuit ($a = 0.5$) to calculate the heat flow into a 6-in. concrete slab with an error of not more than 2 Btu/ft² hr when the surface temperature has an amplitude of 5 F at a frequency of 4 cycles/day, and the other surface is perfectly insulated.

$$\alpha = 0.04 \text{ ft}^2/\text{hr}$$

$$k = 1.0 \text{ Btu/ft hr F}$$

Solution:
$$\phi = \sqrt{\frac{\pi L^2}{\alpha P}} = \sqrt{\frac{\pi \times 0.25}{0.04 \times 6.0}} = 1.8$$

$$R = L/k = 0.5 \text{ ft}^2 \text{ hr F/Btu}.$$

$$\text{when } q_{out} = 0$$

$$\theta_{out} = \frac{1}{A}\,\theta_{in}$$

$$\text{For } \phi = 1.8,\ |1/A| \doteq 0.3$$

$$\text{Thus } \frac{|\theta_{out}|}{|\theta_{in}|} \doteq 0.3$$

The maximum error in the surface heat flux is

$$|q_{in} - q_0|_{max} = \frac{|\theta_{in}|}{R}\left\{U_N + V_N \frac{|\theta_{out}|}{|\theta_{in}|}\right\}$$

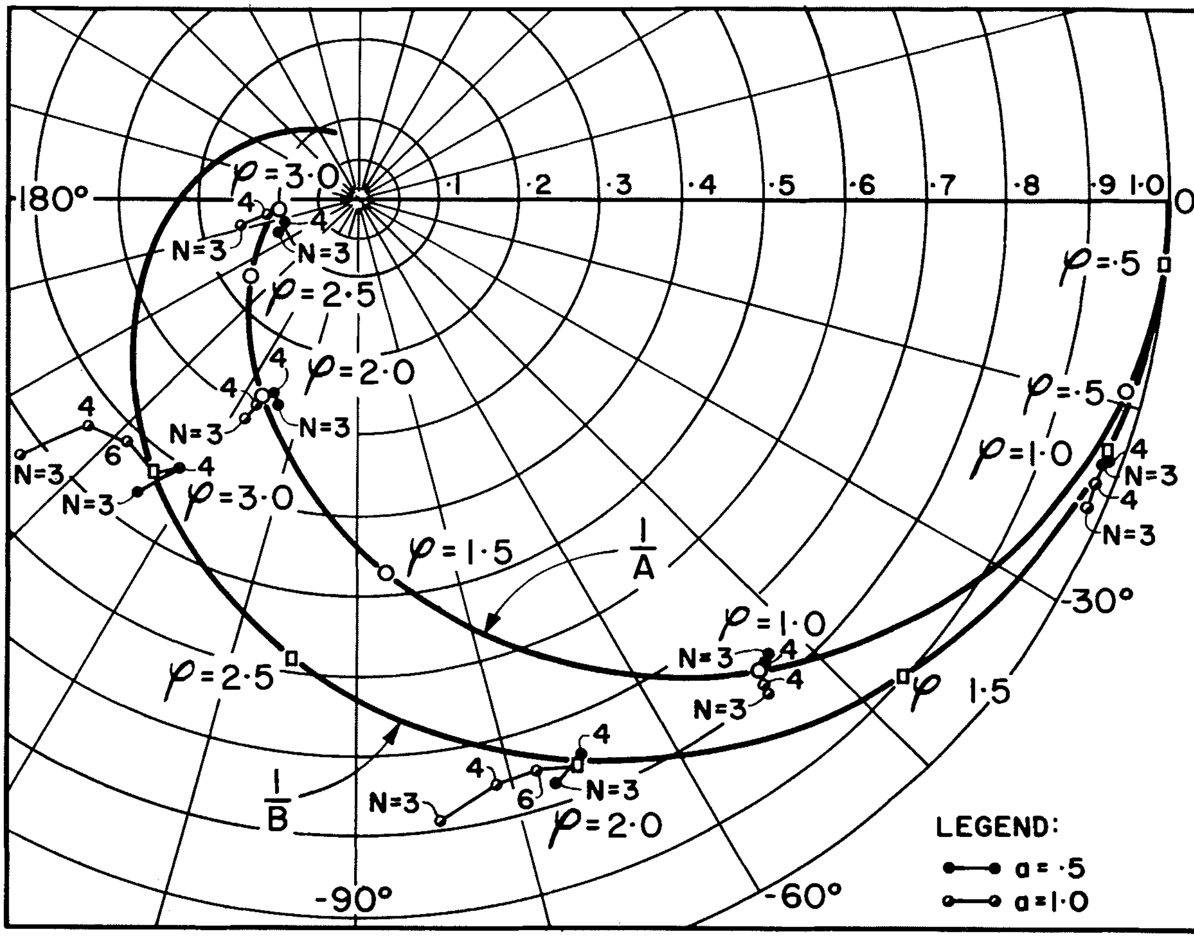

FIG. 4a POLAR COORDINATE PLOT OF $1/A_N$ AND $1/B_N$ FOR AN OPERATIONAL AMPLIFIER CIRCUIT WITH $a = 0.5$ AND $a = 1.0$.

Thus if the error is to be less than 2 Btu/ft² hr

$$U_N + 0.3\, V_N < \frac{R(2)}{|\theta_{in}|} = 0.2$$

The following values of U_N and V_N are taken from Table IV for $\phi = 2.0$. Values for $\phi = 1.8$ can be obtained by interpolation but the nearest plotted value of ϕ usually will suffice to find the required N

$U_3 = 0.32 \qquad V_3 = 0.04$

$U_4 = 0.12 \qquad V_4 = 0.01$

Thus a 4-lump circuit meets the requirement.

If the important results of an analog computation are the temperatures at different points through a slab rather than surface heat flux the error in the temperatures may be estimated as follows:

The temperature at plane x in Fig. 1 is:

$$\theta_x = \left(A_x - \frac{x}{L} \cdot \frac{A\,B_x}{B}\right)\theta_{in} + \left(\frac{x}{L} \cdot \frac{B_x}{B}\right)\theta_{out} \qquad (21)$$

where the matrix elements with subscript x are for $\phi_x = x\sqrt{\frac{\pi}{\alpha P}}$ and the quantities without subscripts are for $\phi = L\sqrt{\frac{\pi}{\alpha P}}$. For an analog circuit with $a = 1.0$

$$\theta_i = \left(A_i - \frac{i}{N} \cdot \frac{A_N B_i}{B_N}\right)\theta_{in} + \left(\frac{i}{N} \cdot \frac{B_i}{B_N}\right)\theta_{out} \qquad (22)$$

where A_i and B_i are matrix elements for an i lump analog circuit with $\phi_i = \frac{i}{N}\,\phi$. For values of a, other than unity, the A_i and B_i values cannot be taken from the tables but can be calculated by the method given in Appendix II. Thus the error in the analog temperature for the plane x_i is

$$\theta_x - \theta_i = \left\{A_x - A_i + \frac{x}{L}\left(\frac{A_N\,B_i}{B_N} - \frac{A\,B_x}{B}\right)\right\}\theta_{in} + \frac{x}{L}\left\{\frac{B_x}{B} - \frac{B_i}{B_N}\right\}\theta_{out} \qquad (23)$$

Since these coefficients are functions of x/L as well as N and ϕ, it seems impractical to prepare tables of the type given for the heat flux errors. The values of the separate factors can be obtained from Table I. The second example illustrates the procedure.

Example 2.

Problem: Find the maximum possible error in the temperature at $x = 0.75\ L$ for a 6-in. concrete slab which is represented by a four lump operational amplifier analog circuit, $a = 1.0$. Properties and boundary conditions same as for example 1.

Solution: $\phi = 1.8$ but use 2.0 to avoid need for interpolation of the tables.

Data:

$$A_{.75L} = \frac{1}{.4694} \angle 85.5$$

$$B_{.75L} = \frac{1}{.9022} \angle 41.3$$

$$A_3 = \frac{1}{.4919} \angle 86.7$$

$$B_3 = \frac{1}{.9094} \angle 50.1$$

$$A = \frac{1}{.2739} \angle 115.4$$

$$B = \frac{1}{.7565} \angle 68.6$$

$$A_4 = \frac{1}{.2831} \angle 115.9$$

$$B_4 = \frac{1}{.7526} \angle 76.9$$

These give

$$\left|\theta_{.75L} - \theta_3\right|_{MAX} = 0.01\,F$$

and

$$\left|\theta_{.75L}\right| = 0.78\,F$$

Figure 5 shows that, in general, a resistance-capacitance analog circuit requires more lumps than an operational amplifier analog circuit to achieve the same accuracy for heat flow calculations. An

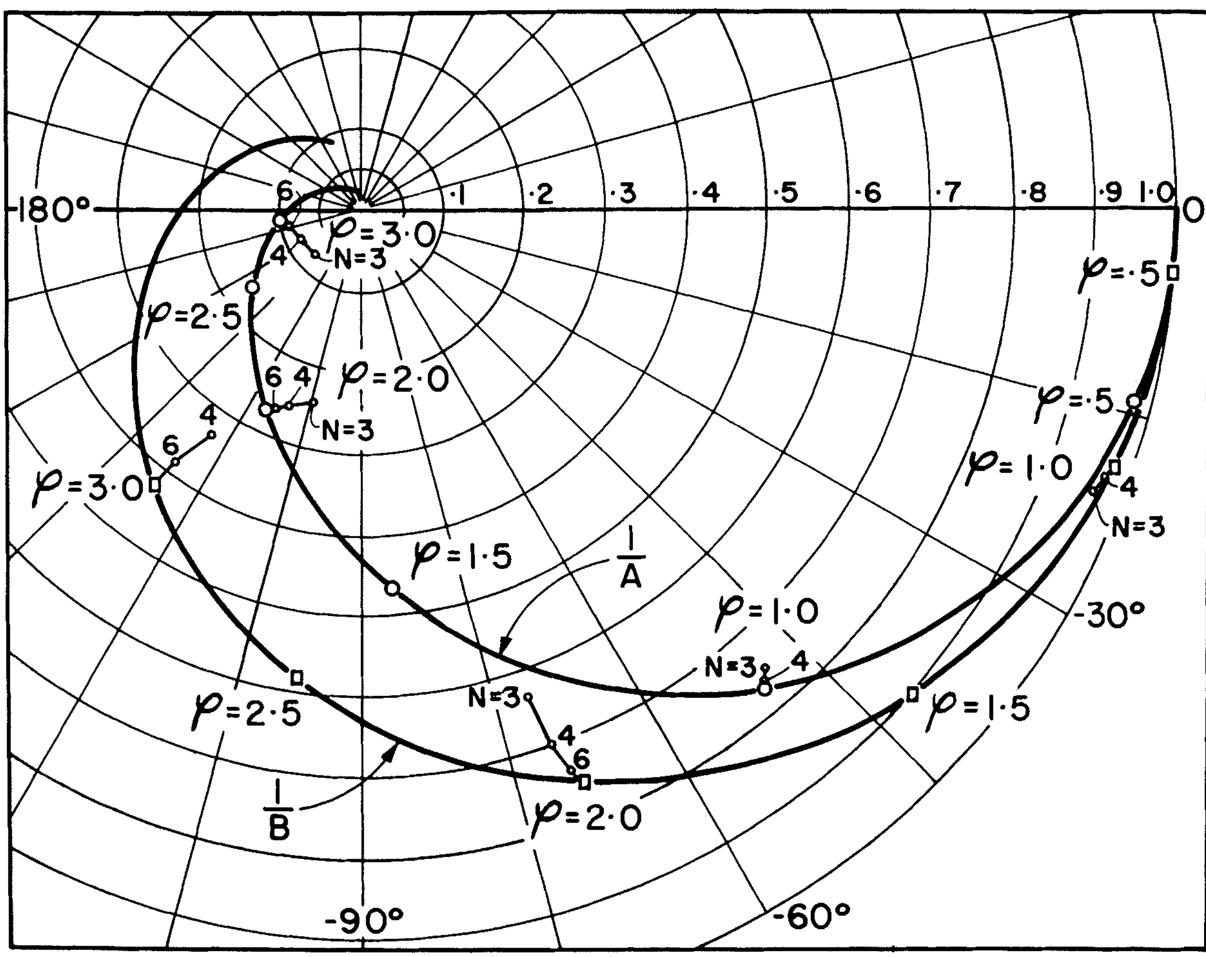

FIG. 4b POLAR COORDINATE PLOT OF $1/A_N$ AND $1/B_N$ FOR A RESISTANCE-CAPACITANCE LADDER NETWORK WITH $a = 0.5$.

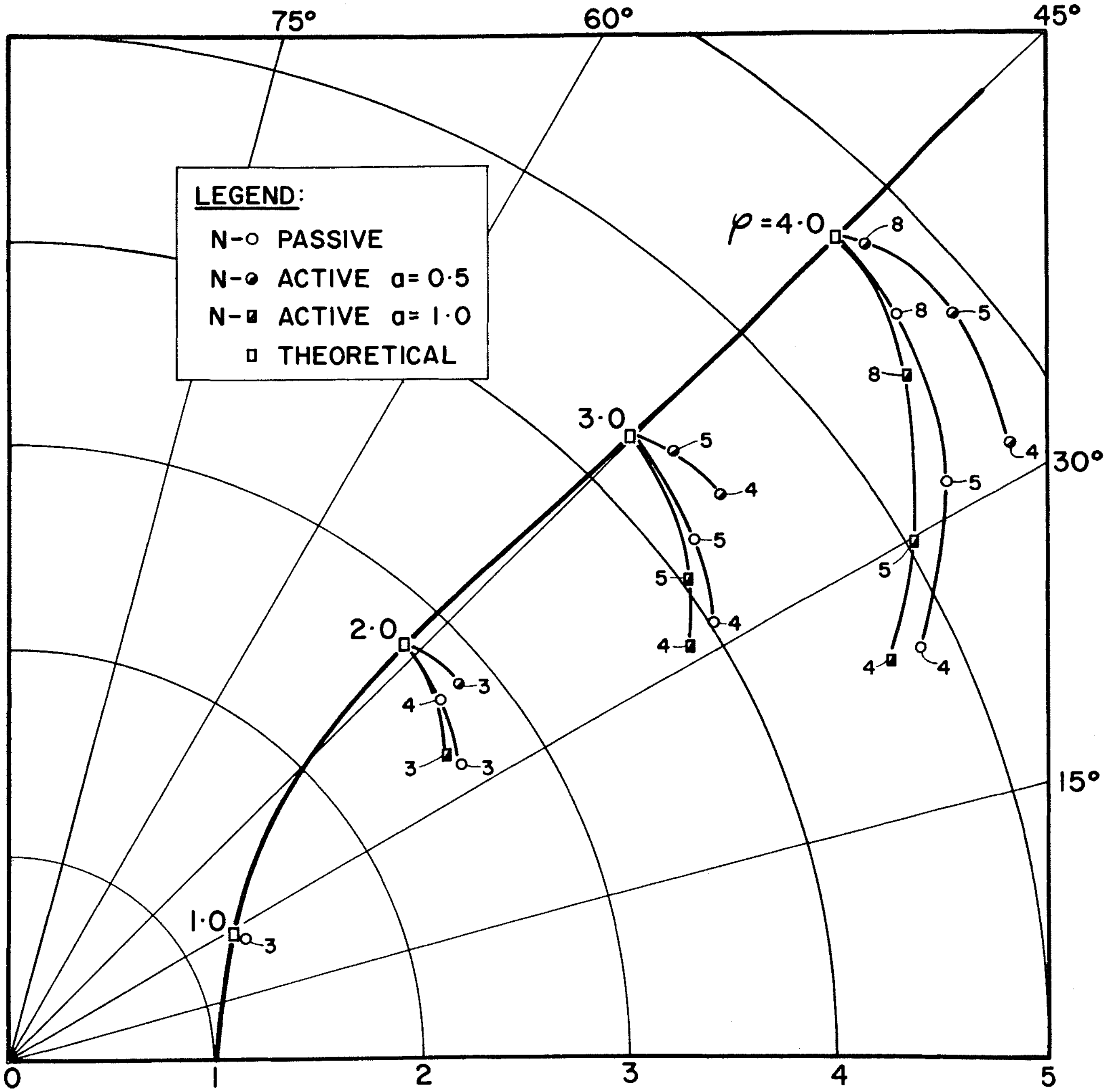

FIG. 5 POLAR COORDINATE PLOT OF A_N/B_N FOR A RESISTANCE-CAPACITANCE LADDER NETWORK WITH $a = 0.5$ AND AN OPERATIONAL AMPLIFIER CURCUIT WITH $a = 0.5$ AND $a = 1.0$.

N lump circuit of either type, however, will give exactly the same temperatures at the internal points. It should be possible, therefore, to use a hybrid analog which will have the same accuracy as the operational amplifier type. It would use amplifier adders to calculate the heat fluxes and a less expensive resistance-capacitance network to compute the temperatures.

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REFERENCES

1. Mitalas, G.P., Stephenson, D.G., and Baxter, D.C., "Use of an analog computer for room air conditioning calculations." Proceedings of the

Second Conference of the Computing and Data Processing Society of Canada, held in Toronto, June 1960, pp. 175-192 (NRC6099).

2. Paschkis, V. and Heisler, M.P., "The accuracy of measurements in lumped R-C cable circuits as used in the study of transient heat flow," *Trans. Amer. Inst. Elect. Engineers*, Vol. 63, 1944, p. 165.

3. Klein, E.O.P., Touloukian, Y.S., and Eaton, J.R., "Limits of accuracy of electrical analog circuits used in the solution of transient heat conduction problems," presented at the Annual Meeting of the Amer. Soc. of Mech. Engineers, 1952 (ASME Paper No. 52-A-65).

4. Lawson, D.I. and McGuire, J.H., "The solution of transient heat flow problems by analogous electrical networks," *Proceedings, Institution of Mechanical Engineers*, (A), Vol. 167, 1953, p. 275.

5. Friedmann, N.E., "The truncation error in a semi-discrete analog of the heat equation," *Journal of Math. and Phys*, Vol. 35, No. 3, 1956, p. 299.

6. Carslaw, H.S. and J.C. Jaeger, *Conduction of heat in solids*, Oxford University Press, Second Edition, 1959.

7. Shirtliffe, C.J. and Stephenson, D.G., Tables of cosh $(1+i)X$, $\frac{\sinh(1+i)X}{(1+i)X}$ and $(1+i)X \sinh(1+i)X$. National Research Council, Division of Building Research, Ottawa, January 1961, 71 pp. (NRC6159).

APPENDIX I: FINITE DIFFERENCE EXPRESSIONS FOR SPACE DERIVATIVES

Let subscript 0 indicate $x = 0$, subscript 1 indicate $x = a \cdot \Delta x$, and subscript 2 indicate $x=(1+a)\Delta x$. Roman numeral superscript indicates the order of derivative with respect to x.

A Taylor's series expansion about the point $x=a \cdot \Delta x$ gives

$$\theta_0 = \theta_1 - a\cdot\Delta x\,\theta_1^{\mathrm{I}} + \frac{(a\cdot\Delta x)^2}{2!}\theta_1^{\mathrm{II}} - \frac{(a\cdot\Delta x)^3}{3!}\theta_1^{\mathrm{III}} + \frac{(a\cdot\Delta x)^4}{4!}\theta_1^{\mathrm{IV}} - + \dots\dots \qquad (\mathrm{I.1})$$

and $$\theta_2 = \theta_1 + \Delta x\,\theta_1^{\mathrm{I}} + \frac{(\Delta x)^2}{2!}\theta_1^{\mathrm{II}} + \frac{(\Delta x)^3}{3!}\theta_1^{\mathrm{III}} + \frac{(\Delta x)^4}{4!}\theta_1^{\mathrm{IV}} + \dots\dots \qquad (\mathrm{I.2})$$

Multiplying I.2 by a and adding to I.1 gives

$$\theta_0 + a\theta_2 = (1+a)\theta_1 + \frac{a(1+a)}{2}(\Delta x)^2\theta_1^{\mathrm{II}} + \frac{a(1-a^2)}{6}(\Delta x)^3\theta_1^{\mathrm{III}} + \frac{a(1+a^3)(\Delta x)^4}{24}\theta_1^{\mathrm{IV}} + \dots\dots \qquad (\mathrm{I.3})$$

$$(\Delta x)^2\theta_1^{\mathrm{II}} = \frac{2}{a(1+a)}\theta_0 - \frac{2}{a}\theta_1 + \frac{2}{1+a}\theta_2 - \frac{(1-a)}{3}(\Delta x)^3\theta_1^{\mathrm{III}} - \frac{1+a^3}{12(1+a)}(\Delta x)^4\theta_1^{\mathrm{IV}} - \dots\dots \qquad (\mathrm{I.4})$$

For the other internal points $(1 < i < N-1)$ the second derivative expression is similar to (I.4) except that $a = 1$, i.e.

$$(\Delta x)^2\theta_i^{\mathrm{II}} = \theta_{i-1} - 2\theta_i + \theta_{i+1} - \frac{(\Delta x)^4}{12}\theta_i^{\mathrm{IV}} \qquad (\mathrm{I.5})$$

Differentiating (I.1) gives

$$\theta_0^{\mathrm{I}} = \theta_1^{\mathrm{I}} - (a\Delta x)\,\theta_1^{\mathrm{II}} + \frac{(a\cdot\Delta x)^2}{2!}\theta_1^{\mathrm{III}} - \frac{(a\cdot\Delta x)^3}{3!}\theta_1^{\mathrm{IV}} \qquad (\mathrm{I.6})$$

Equations (I.1) and (I.2) give

$$\Delta x\,\theta_1^{\mathrm{I}} = -\frac{1}{a(1+a)}\theta_0 + \frac{1-a}{a}\theta_1 + \frac{a}{1+a}\theta_2 - \frac{a}{6}(\Delta x)^3\theta_1^{\mathrm{III}} - \frac{a(1-a)}{24}(\Delta x)^4\theta_1^{\mathrm{IV}} \qquad (\mathrm{I.7})$$

Multiplying (I.6) by Δx and substituting (I.7) and (I.4) for $\Delta x\,\theta_1^{\mathrm{I}}$ and $(\Delta x)^2\theta_1^{\mathrm{II}}$ respectively gives

$$(\Delta x)\,\theta_0^{\mathrm{I}} = -\frac{(1+2a)}{a(1+a)}\theta_0 + \frac{1+a}{a}\theta_1 - \frac{a}{1+a}\theta_2 + \frac{a(1+a)}{6}(\Delta x)^3\theta_1^{\mathrm{III}} + \frac{a(1+a)(1-2a)}{24}(\Delta x)^4\theta_1^{\mathrm{IV}} \qquad (\mathrm{I.8})$$

when $a = 1.0$

$$\theta_1^{\mathrm{II}} = \frac{\theta_0 - 2\theta_1 + \theta_2}{(\Delta x)^2} + 0\,\theta_1^{\mathrm{III}} - \frac{(\Delta x)^2}{12}\theta_1^{\mathrm{IV}} \qquad (\mathrm{I.9})$$

$$\theta_0^{\mathrm{I}} = \frac{-3\theta_0 + 4\theta_1 - \theta_2}{2\Delta x} + \frac{(\Delta x)^2}{3}\theta_1^{\mathrm{III}} - \frac{(\Delta x)^3}{12}\theta_1^{\mathrm{IV}} \qquad (\mathrm{I.10})$$

when $a = 0.5$

$$\theta_1^{\text{II}} = \frac{8\,\theta_0 - 12\,\theta_1 + 4\,\theta_2}{3(\Delta x)^2} - \frac{\Delta x}{6}\,\theta_1^{\text{III}} - \frac{(\Delta x)^2}{16}\,\theta_1^{\text{IV}} \quad \text{(I.11)}$$

$$\theta_0^{\text{I}} = \frac{-8\,\theta_0 + 9\,\theta_1 - \theta_2}{3\Delta x} + \frac{(\Delta x)^2}{8}\,\theta_1^{\text{III}} + 0\,(\theta_1^{\text{IV}}) \quad \text{(I.12)}$$

APPENDIX II: CALCULATION OF TRANSFER FUNCTIONS FOR AN OPERATIONAL AMPLIFIER TYPE ANALOG CIRCUIT

The sinusoidal temperature and heat flow at the two surfaces of a slab are given by equation (1). A similar expression relates the voltages in an analog circuit which simulates the heat flow and temperature in a slab.

$$\begin{bmatrix} V_{\theta_0} \\ V_{q_0} \end{bmatrix} = \begin{bmatrix} A_N, & R\,B_N \\ \frac{D_N}{R}, & A_N \end{bmatrix} \cdot \begin{bmatrix} V_{\theta_N} \\ V_{q_N} \end{bmatrix} \quad \text{(II.1)}$$

The subscript on the V's indicates the thermal quantity which the voltage represents. For simplicity in notation θ_0 is subsequently used in place of V_{θ_0} and similarly for the other quantities with the understanding that it refers to the voltage when used in connection with an analog circuit. N is the total number of lumps that are used in the circuit so A_N and B_N are the transmission matrix coefficients for an N lump analog circuit. A subscript i on the temperatures and heat flows indicates that the quantity pertains to the output of the i^{th} lump. Thus the subscripts 0 and N represent the two surfaces.

Case 1. All Lumps the Same. This is the case where $a = 1.0$. Equations (5) and (6) become

$$-\left(\frac{\partial\theta}{\partial x}\right)_0 = \frac{3\theta_0 - 4\theta_1 + \theta_2}{2\,\Delta x} \quad \text{(II.2)}$$

$$\left(\frac{\partial^2\theta}{\partial x^2}\right)_i = \frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{(\Delta x)^2} \quad \text{(II.3)}$$

for $1 \leq i \leq N-1$

and

$$-\left(\frac{\partial\theta}{\partial x}\right)_N = \frac{4\theta_{N-1} - 3\theta_N - \theta_{N-2}}{2\,\Delta x} \quad \text{(II.4)}$$

$$\Delta x = L/N \quad \text{(II.5)}$$

Hence

$$q_N = \frac{k}{\Delta x}\left(\frac{-3\theta_N + 4\theta_{N-1} - \theta_{N-2}}{2}\right) \quad \text{(II.6)}$$

$$\text{or } Rq_N = \frac{N}{2}\,(-3\theta_N + 4\theta_{N-1} - \theta_{N-2}) \quad \text{(II.7)}$$

$$\text{and } \frac{d}{dt}(\theta_i) = \frac{a}{(\Delta x)^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) \quad \text{(II.8)}$$

for $1 \leq i \leq N-1$

Expression for A_N.

When $q_N = 0 \qquad \theta_0 = A_N\,\theta_N$

$$\text{and } 4\,\theta_{N-1} = 3\theta_N + \theta_{N-2} \quad \text{(II.9)}$$

Let $\dfrac{a}{(\Delta x)^2} = 1$

$$\eta_i = \mathfrak{L}\{\theta_i\} = \int_0^\infty e^{-st}\cdot\theta_i\,dt$$

Transforming equations (II.8) for $i = N-1$ and (II.9) gives

$$(s+2)\;\eta_{N-1} = \eta_N + \eta_{N-2} \quad \text{(II.10)}$$

$$4\,\eta_{N-1} = 3\,\eta_N + \eta_{N-2} \quad \text{(II.11)}$$

Let $s + 2 = y$

$$\text{Then } \frac{\eta_{N-1}}{\eta_N} = \frac{2}{4-y} \quad \text{(II.12)}$$

$$\frac{\eta_{N-2}}{\eta_N} = y\left(\frac{2}{4-y}\right) - 1 = \frac{3y-4}{4-y} \quad \text{(II.13)}$$

The following general recurrence relationship holds for all other points

$$\frac{\eta_{N-i}}{\eta_N} = y\left(\frac{\eta_{N+1-i}}{\eta_N}\right) - \left(\frac{\eta_{N+2-i}}{\eta_N}\right) \quad \text{(II.14)}$$

for $2 \leq i \leq N$

For each value of N the ratio

$$\frac{\eta_0}{\eta_N} = \mathfrak{L}\{A_N\} \quad \text{(II.15)}$$

To get the steady periodic response of a system to a sinusoidal driving function only requires substituting $j\ w$ for s everywhere in the expression for the Laplace transform of the response, where $w = 2\pi/P$. Since $\alpha/(\Delta x)^2$ has been assumed equal to unity

$$w = 2\left(\frac{\phi}{N}\right)^2 \qquad \text{(II.16)}$$

For example: for $N = 3$

$$\mathfrak{L}\{A_3\} = y\left(\frac{3y-4}{4-y}\right) - \frac{2}{4-y}$$

$$= \frac{3y^2 - 4y - 2}{4-y} \qquad \text{(II.17)}$$

$$\text{Thus } A_3 = \frac{3Z^2 - 4Z - 2}{4 - Z} \qquad \text{(II.18)}$$

$$\text{Where } Z = 2 + jw = 2\left[1 + j\left(\frac{\phi}{3}\right)^2\right] \qquad \text{(II.19)}$$

Expression for B_N.

When $\theta_N = 0 \qquad \theta_0 = B_N\ (Rq_N)$

$$Rq_N = \frac{N}{2}\ (4\theta_{N-1} - \theta_{N-2}) \qquad \text{(II.20)}$$

and

$$\frac{d}{dt}(\theta_{N-1}) = \frac{\alpha}{(\Delta x)^2}\ (\theta_{N-2} - 2\,\theta_{N-1}) \qquad \text{(II.21)}$$

The differential equations for the other temperatures are given by II.8 for $1 \leq i \leq N-2$.

Again let $\alpha/\Delta x)^2 = 1$ and take Laplace transforms of II.20 and II.21. This gives:

$$\mathfrak{L}\{Rq_N\} = \frac{N}{2}\ (4\,\eta_{N-1} - \eta_{N-2}) \qquad \text{(II.22)}$$

$$\eta_{N-2} = y\ \eta_{N-1} \qquad \text{(II.23)}$$

Hence

$$\frac{\eta_{N-1}}{\mathfrak{L}\{Rq_N\}} = \frac{2}{N(4-y)} \qquad \text{(II.24)}$$

and

$$\frac{\eta_{N-2}}{\mathfrak{L}\{Rq_N\}} = \frac{2y}{N(4-y)} \qquad \text{(II.25)}$$

The transform of equation (II.8) gives the general recurrence relationship for

$$3 \leq i \leq N$$

$$\frac{\eta_{N-i}}{\mathfrak{L}\{Rq_N\}} = y\left(\frac{\eta_{N+1-i}}{\mathfrak{L}\{Rq_N\}}\right) - \frac{\eta_{N+2-i}}{\mathfrak{L}\{Rq_N\}} \qquad \text{(II.26)}$$

For each N the ratio $\eta_0 / \ \mathfrak{L}\{Rq_N\} = \mathfrak{L}\{B_N\}$

For example: for $N = 3$

$$\mathfrak{L}\{B_3\} = \frac{2\,y^2 - 2}{3\ (4-y)} \qquad \text{(II.27)}$$

The inversion of this is just the same as for the A_N terms i.e.

$$B_3 = \frac{2\,Z^2 - 2}{3\ (4-Z)} \qquad \text{(II.28)}$$

$$\text{where} \qquad Z = 2\left[1 + j\left(\frac{\phi}{3}\right)^2\right] \qquad \text{(II.29)}$$

Case 2. Half Lumps at the Surfaces. This is the situation where $a = 0.5$. The finite difference expressions become

$$-\left(\frac{\partial\theta}{\partial x}\right)_N = \frac{8\,\theta_N - 9\,\theta_{N-1} + \theta_{N-2}}{3\,\Delta x}$$

$$\left(\frac{\partial^2\theta}{\partial x^2}\right)_{N-1} = \frac{8\,\theta_N - 12\,\theta_{N-1} + 4\,\theta_{N-2}}{3\ (\Delta x)^2}$$

$$\left(\frac{\partial^2\theta}{\partial x^2}\right)_i = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta x)^2}$$

for $1 < i < N-1$

and

$$\left(\frac{\partial^2\theta}{\partial x^2}\right)_1 = \frac{8\,\theta_0 - 12\,\theta_1 + 4\,\theta_2}{3(\Delta x)^2}$$

$$\Delta x = \frac{L}{N-1}$$

The development of the expressions for A_N and B_N for this case is similar to case 1 except that the final step is

$$\eta_0 = \frac{3s+12}{8}\ \eta_1 - \tfrac{1}{2}\eta_2$$

rather than the standard recurrence formula; and $w = 2\left(\frac{\phi}{N-1}\right)^2$.

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LUMPING ERRORS OF ANALOG CIRCUITS FOR HEAT FLOW THROUGH A HOMOGENEOUS SLAB

by D. G. Stephenson and G. P. Mitalas

TABLES I - IV

National Research Council, Canada
Division of Building Research

TABLE I

VALUES OF $1/A_N$ AND $1/B_N$ AS A FUNCTION OF ϕ AND N FOR AN OPERATIONAL AMPLIFIER ANALOG CIRCUIT WITH a = 1.0

ϕ	N = 3		N = 4		N = 5		N = 6		Theoretical	
0.5	.9809	- 14.2	.9801	- 14.2	.9799	- 14.2	.9798	- 14.2	.9798	- 14.2
	.9987	- 5.8	.9985	- 5.4	.9985	- 5.2	.9985	- 5.0	.9986	- 4.8
1.0	.7839	- 50.2	.7761	- 50.0	.7739	- 49.9	.7732	- 49.9	.7731	- 49.9
	.9795	- 23.1	.9773	- 21.5	.9772	- 20.4	.9774	- 20.0	.9785	- 18.9
1.5	.4919	- 86.7	.4756	- 85.9	.4711	- 85.6	.4696	- 85.5	.4694	- 85.5
	.9094	- 50.1	.8984	- 46.4	.8974	- 44.6	.8980	- 43.6	.9022	- 41.3
2.0	.3074	-116.9	.2831	-115.9	.2766	-115.5	.2745	-115.3	.2739	-115.4
	.7842	- 82.1	.7526	- 76.9	.7478	- 74.1	.7480	- 72.5	.7565	- 68.8
2.5	.2074	-143.6	.1761	-143.8	.1676	-143.5	.1648	-143.3	.1638	-143.6
	.6456	-113.9	.5848	-108.7	.5725	-105.2	.5708	-103.1	.5815	- 97.9
3.0	.1499	-167.0	.1139	-170.6	.1040	-171.1	.1006	-171.1	.0993	-171.9
	.5262	-142.1	.4386	-139.4	.4177	-136.0	.4130	-133.5	.4235	-126.8
3.5	.1136	-186.6	.0760	-195.9	.0656	-198.2	.0619	-198.8	.0604	-200.5
	.4313	-165.7	.3260	-168.1	.2982	-165.8	.2906	-163.4	.2991	-155.6
4.0	.0890	-202.3	.0521	-218.7	.0419	-224.1	.0384	-225.9	.0366	-229.2
	.3561	-184.8	.2425	-194.0	.2110	-194.3	.2012	-192.7	.2072	-184.2
5.0	.0585	-224.0	.0261	-255.7	.0180	-270.5	.0152	-277.0	.0135	-286.5
	.2485	-211.6	.1359	-236.6	.1049	-245.9	.0941	-248.1	.0353	-241.5
6.0	.0411	-237.3	.0141	-282.2	.0082	-308.4	.0062	-322.3	.0050	-343.8
	.1796	-228.3	.0780	-267.5	.0525	-288.3	.0436	-297.7	.0421	-298.8
7.0	.0304	-245.8	.0081	-300.7	.0039	-337.9	.0027	-360.8	.0018	-401.1
	.1345	-238.9	.0464	-289.2	.0267	-321.5	.0202	-339.9	.0181	-356.1

TABLE I

(Continued)

ϕ	N = 8		N = 10		N = 15		N = 20		Theoretical	
1.0	.7728	- 49.8	.7728	- 49.9	.7729	- 49.9	.7730	- 49.9	.7731	- 49.9
	.9777	- 19.5	.9780	- 19.3	.9782	- 19.1	.9783	- 19.0	.9785	- 18.9
2.0	.2734	-115.2	.2732	-115.2	.2734	-115.3	.2736	-115.3	.2739	-115.4
	.7503	- 70.9	.7521	- 70.2	.7543	- 69.4	.7552	- 69.1	.7565	- 68.8
3.0	.0988	-171.2	.0986	-171.3	.0988	-171.6	.0990	-171.7	.0993	-171.9
	.4138	-130.8	.4161	-129.4	.4196	-128.0	.4212	-127.5	.4235	-126.8
4.0	.0364	-227.1	.0360	-227.6	.0361	-228.3	.0363	-228.6	.0366	-229.2
	.1983	-189.7	.1997	-187.8	.2030	-185.9	.2047	-185.1	.2072	-184.2
5.0	.0135	-281.6	.0131	-283.2	.0131	-284.7	.0132	-285.4	.0135	-286.5
	.0891	-247.3	.0894	-245.7	.0917	-243.5	.0931	-242.6	.0953	-241.5
6.0	.0050	-333.6	.0048	-337.4	.0048	-340.7	.0048	-341.9	.0050	-343.8
	.0387	-302.5	.0382	-302.4	.0395	-300.8	.0404	-300.0	.0421	-298.8
7.0	.0019	-382.0	.0017	-389.8	.0017	-395.9	.0017	-398.0	.0018	-401.1
	.0165	-354.2	.0159	-357.4	.0164	-357.6	.0170	-357.1	.0181	-356.1
8.0	.0007	-426.0	.0006	-439.7	.0005	-450.4	.0006	-453.7	.0007	-458.4
	.0070	-401.3	.0065	-409.9	.0066	-413.7	.0069	-413.6	.0076	-413.4
9.0	.0003	-465.0	.0002	-486.4	.0002	-503.8	.0002	-508.8	.0002	-515.7
	.0029	-443.3	.0026	-459.3	.0026	-468.8	.0028	-470.1	.0031	-470.7
10.0	.0001	-499.0	.0001	-529.7	.0001	-555.9	.0001	-563.4	.0001	-573.0
	.0013	-479.8	.0010	-505.0	.0010	-522.6	.0011	-525.8	.0013	-528.0

NOTE: The entries in the tables are the quantities shown at right. Theoretical is same as $N = \infty$

	N	
ϕ	$1/\lvert A_N \rvert$	$-\angle A_N$
	$1/\lvert B_N \rvert$	$-\angle B_N$

TABLE II

VALUES OF $1/A_N$ AND $1/B_N$ AS A FUNCTION OF ϕ AND N FOR AN OPERATIONAL AMPLIFIER ANALOG CIRCUIT WITH a = 0.5

ϕ	N = 3		N = 4		N = 5		N = 6		Theoretical	
0.5	.9792	- 14.2	.9792	- 14.2	.9794	- 14.2	.9795	- 14.2	.9798	- 14.2
	.9988	- 4.9	.9986	- 4.8	.9985	- 4.8	.9986	- 4.8	.9986	- 4.8
1.0	.7684	- 49.5	.7684	- 49.6	.7698	- 49.7	.7707	- 49.8	.7731	- 49.9
	.9805	- 19.6	.9775	- 18.9	.9773	- 18.9	.9775	- 18.9	.9785	- 18.9
1.5	.4645	- 84.0	.4621	- 84.5	.4639	- 84.8	.4654	- 85.0	.4694	- 85.5
	.9104	- 42.8	.8983	- 41.2	.8974	- 41.1	.8981	- 41.1	.9022	- 41.3
2.0	.2725	-111.6	.2661	-112.8	.2676	-113.7	.2692	-114.2	.2739	-115.4
	.7721	- 71.5	.7477	- 68.4	.7459	- 68.1	.7473	- 68.2	.7565	- 68.8
2.5	.1668	-135.8	.1562	-138.6	.1570	-140.3	.1586	-141.4	.1638	-143.6
	.6013	-101.2	.5681	- 96.8	.5656	- 96.4	.5677	- 96.5	.5815	- 97.9
3.0	.1071	-157.4	.0928	-162.9	.0927	-166.1	.0940	-168.0	.0993	-171.9
	.4470	-128.9	.4073	-124.3	.4041	-124.0	.4065	-124.4	.4235	-126.8
3.5	.0724	-176.1	.0557	-185.5	.0547	-191.0	.0555	-194.1	.0604	-200.5
	.3301	-152.9	.2829	-150.3	.2785	-150.6	.2808	-151.5	.2991	-155.6
4.0	.0515	-191.9	.0340	-206.3	.0322	-214.6	.0326	-219.3	.0366	-229.2
	.2481	-172.9	.1934	-174.5	.1874	-176.2	.1891	-177.9	.2072	-184.2
5.0	.0298	-215.2	.0136	-241.3	.0113	-257.0	.0111	-266.4	.0135	-286.5
	.1519	-202.2	.0904	-216.5	.0813	-223.4	.0811	-228.0	.0953	-241.5
6.0	.0197	-230.4	.0062	-268.2	.0042	-292.5	.0038	-308.2	.0050	-343.8
	.1025	-221.1	.0445	-249.1	.0347	-264.3	.0332	-273.6	.0421	-298.8
7.0	.0141	-240.3	.0032	-288.3	.0017	-321.5	.0014	-344.3	.0018	-401.1
	.0741	-233.4	.0238	-273.5	.0153	-298.2	.0134	-313.9	.0181	-356.1

TABLE II

(Continued)

ϕ	N = 8		N = 10		N = 15		N = 20		Theoretical	
1.0	.7717	- 49.8	.7722	- 49.8	.7727	- 49.8	.7729	- 49.9	.7731	- 49.9
	.9778	- 18.9	.9780	- 18.9	.9782	- 18.9	.9783	- 18.9	.9785	- 18.9
2.0	.2711	-114.7	.2721	-115.0	.2731	-115.2	.2735	-115.3	.2739	-115.4
	.7503	- 68.4	.7522	- 68.5	.7544	- 68.7	.7553	- 68.7	.7565	- 68.8
3.0	.0960	-169.8	.0971	-170.6	.0983	-171.3	.0988	-171.6	.0393	-171.9
	.4119	-125.2	.4155	-125.8	.4196	-126.3	.4212	-126.5	.4235	-126.8
4.0	.0339	-223.9	.0348	-225.9	.0358	-227.7	.0361	-228.4	.0366	-229.2
	.1944	-180.2	.1982	-181.5	.2028	-182.9	.2046	-183.5	.2072	-184.2
5.0	.0117	-275.8	.0122	-279.9	.0129	-283.7	.0131	-284.9	.0135	-286.5
	.0845	-233.3	.0875	-236.1	.0914	-239.0	.0930	-240.1	.0953	-241.5
6.0	.0040	-324.8	.0042	-332.1	.0046	-338.9	.0047	-341.1	.0050	-343.8
	.0346	-284.0	.0365	-289.2	.0392	-294.4	.0403	-296.3	.0421	-298.8
7.0	.0013	-370.1	.0014	-382.1	.0016	-393.2	.0017	-396.8	.0018	-401.1
	.0136	-331.6	.0145	-340.4	.0161	-349.1	.0169	-352.1	.0181	-356.1
8.0	.0004	-411.3	.0005	-429.4	.0006	-446.4	.0006	-451.9	.0007	-458.4
	.0052	-375.6	.0056	-389.2	.0064	-402.8	.0069	-407.4	.0076	-413.4
9.0	.0002	-448.2	.0002	-473.7	.0002	-498.4	.0002	-506.3	.0002	-515.7
	.0020	-415.4	.0021	-435.3	.0025	-455.3	.0027	-462.1	.0031	-470.7
10.0	.0001	-480.8	.0001	-514.7	.0001	-548.9	.0001	-560.0	.0001	-573.0
	.0008	-451.0	.0008	-478.3	.0009	-506.5	.0011	-516.1	.0013	-528.0

NOTE: The entries in the tables are the quantities shown at right. Theoretical is same as N = ∞

	N	
ϕ	$1/\lvert A_N \rvert$	$-\angle A_N$
	$1/\lvert B_N \rvert$	$-\angle B_N$

TABLE III

VALUES OF $1/A_N$ AND $1/B_N$ AS A FUNCTION OF ϕ AND N FOR PASSIVE NETWORKS WITH a = 0.5

ϕ	N = 3		N = 4		N = 5		N = 6		Theoretical	
0.5	.977[illegible]	- 14.1	.9767	- 14.2	.9752	- 14.2	.975[illegible]	- 14.2	.975[illegible]	- 14.2
	.9975	- 5.4	.9982	- 5.0	.9984	- 4.9	.9985	- 4.9	.9986	- 4.8
1.0	.7524	- 48.8	.7657	- 49.4	.7677	- 49.6	.7697	- 49.7	.7731	- 49.9
	.9627	- 21.2	.9720	- 19.9	.9749	- 19.5	.9762	- 19.[illegible]	.9765	- 18.9
1.5	.430[illegible]	- 80.7	.4542	- 83.5	.4605	- 84.4	.4636	- 84.[illegible]	.4694	- 85.9
	.[illegible]	- 45.1	.8754	- 43.2	.8879	- 42.4	.8928	- 42.[illegible]	.9022	- 41.3
2.0	.2425	-104.0	.25[illegible]	-110.6	.2[illegible]55	-112.8	.2[illegible]7[illegible]	-11[illegible].7	.2759	-115.4
	.[illegible]325	- 71.6	.[illegible]95	- 7[illegible].7	.7245	- 70.6	.7[illegible]	- [illegible]9.6	.7565	- 68.8
2.5	.1[illegible]	-121.3	.1458	-134.[illegible]	.1524	-1[illegible]	.1561	-14[illegible].4	.1[illegible]	-14[illegible].6
	.4248	- 95.4	.5009	- 98.2	.5[illegible]	- 98.4	.5511	- [illegible]	.5615	- 97.9
3.0	.0750	-155.4	.0828	-155.6	.0881	-16[illegible].1	.0916	-1[illegible]6.4	.0[illegible]	-171.9
	.2[illegible]	-114.4	.3528	-12[illegible].7	.[illegible]	-125.8	.3[illegible]	-126.4	.4255	-126.8
3.5	.0454	-145.4	.0466	-174.2	.0565	-186.1	.0582	-191.[illegible]	.0604	-200.5
	.1697	-128.8	.2111	-14[illegible]	.2417	-151.6	.2[illegible]1	-15[illegible].5	.2581	-155.6
4.0	.0267	-152.7	.0257	-189.9	.0267	-207.5	.0306	-215.[illegible]	.0366	-22[illegible].2
	.1085	-139.4	.1[illegible]05	-165.8	.1534	-175.[illegible]	.1592	-173.5	.2072	-184.2
5.0	.0124	-162.0	.0091	-211.5	.0092	-245.2	.[illegible]	-250.1	.01[illegible]5	-236.5
	.0482	-154.2	.0492	-195.7	.0577	-217.2	.0[illegible]65	-227.[illegible]	.0955	-241.5
6.0	.0061	-167.4	.0034	-229.0	.0030	-270.6	.00[illegible]2	-295.9	.0050	-343.8
	.0240	-161.1	.0135	-215.8	.0211	-249.8	.0241	-2[illegible].8	.0421	-298.8
7.0	.00[illegible]	-170.7	.0014	-239.3	.0011	-2[illegible].7	.0010	-[illegible]	.0018	-401.1
	.0131	-166.1	.00[illegible]	-223.2	.0078	-274.2	.[illegible]07	-[illegible]	.0101	-356.1

TABLE III

(Continued)

ϕ	N = 8		N = 10		N = 12		N = 14		Theoretical	
1.0	.7719	- 49.8	.7720	- 49.8	.7724	- 49.8	.7727	- 49.8	.7731	- 49.9
	.9775	- 19.1	.9778	- 19.1	.9780	- 19.0	.9782	- 19.0	.9785	- 18.9
2.0	.2705	-114.8	.2717	-114.9	.2724	-115.1	.2728	-115.2	.2739	-115.4
	.7461	- 69.2	.7502	- 69.1	.7522	- 69.0	.7534	- 68.9	.7565	- 68.8
3.0	.0951	-169.2	.0967	-170.3	.0975	-170.8	.0980	-171.2	.0993	-171.9
	.4043	-126.7	.4118	-126.8	.4156	-126.8	.4178	-126.8	.4235	-126.8
4.0	.0331	-222.5	.0344	-225.2	.0351	-226.5	.0355	-227.3	.0366	-229.2
	.1862	-182.2	.1941	-185.1	.1985	-186.5	.2008	-187.7	.2072	-184.2
5.0	.0112	-275.1	.0120	-278.7	.0124	-281.3	.0127	-282.8	.0135	-286.9
	.0779	-235.3	.0841	-238.0	.0876	-239.3	.0897	-239.9	.0955	-241.5
6.0	.0037	-320.1	.0041	-330.0	.0043	-334.7	.0045	-337.4	.0050	-343.8
	.0303	-285.2	.0341	-291.2	.0364	-294.0	.0379	-295.4	.0421	-298.0
7.0	.0012	-362.8	.0014	-378.7	.0015	-38[illegible].5	.0016	-390.	.0018	-401.1
	.0111	-331.1	.0131	-342.1	.0145	-347.2	.0153	-350.0	.0181	-356.1
8.0	.0004	-400.6	.0004	-424.5	.0005	-436.2	.0005	-442.8	.0007	-450.4
	.0039	-372.2	.0048	-390.1	.0055	-398.6	.0060	-405.2	.0076	-413.4
9.0	.0001	-433.4	.0001	-466.4	.0002	-485.6	.0002	-491.2	.0002	-515.7
	.0014	-408.2	.0017	-434.7	.0020	-447.8	.0025	-454.9	.0031	-470.7
10.0	.0000	-451.3	.0000	-504.8	.0001	-528.[illegible]	.0001	-541.7	.0001	-575.0
	.0005	-439.1	.0006	-475.7	.0007	-494.5	.0008	-504.9	.0015	-523.0

NOTE: The entries in the tables are the quantities shown at right. Theoretical is same as N = ∞

ϕ	N	
	$1/\lvert A_N \rvert$	$-\angle A_N$
	$1/\lvert B_N \rvert$	$-\angle B_N$

TABLE IV

VALUES OF U_N AND V_N AS A FUNCTION OF ϕ AND N
FOR OPERATIONAL AMPLIFIER AND PASSIVE NETWORK ANALOG CIRCUITS.

OPERATIONAL AMPLIFIER ANALOG												RESISTANCE-CAPACITANCE NETWORK					
a = 0.5						a = 1.0						a = 0.5					
N=3	N=4	N=5	N=6	N=8	N=10	N=3	N=4	N=5	N=6	N=8	N=10	N=3	N=4	N=5	N=6	N=8	N=10
.01 .00	.00 .00	.00 .00	.00 .00	.00 .00	.00 .00	.03 .01	.01 .00	.00 .00	.00 .00	.00 .00	.00 .00	.02 .01	.00 .00	.00 .00	.00 .00	.00 .00	.00 .00
.04 .01	.01 .00	.00 .00	.00 .00	.00 .00	.00 .00	.09 .07	.05 .04	.03 .03	.01 .02	.00 .01	.00 .01	.08 .04	.03 .02	.01 .01	.00 .00	.00 .00	.00 .00
.14 .02	.08 .00	.04 .00	.02 .00	.00 .00	.00 .00	.26 .13	.15 .08	.11 .05	.03 .03	.02 .02	.00 .01	.30 .08	.14 .04	.08 .02	.05 .01	.06 .00	.00 .00
.32 .04	.12 .01	.06 .01	.03 .01	.00 .01	.00 .00	.57 .18	.37 .10	.25 .07	.19 .05	.11 .03	.08 .02	.65 .13	.33 .06	.19 .03	.13 .02	.06 .01	.04 .00
.71 .04	.27 .02	.13 .02	.05 .02	.01 .01	.00 .00	1.03 .17	.69 .10	.46 .07	.36 .04	.20 .02	.15 .02	1.20 .16	.60 .08	.34 .05	.25 .03	.13 .02	.07 .01
.25 .03	.52 .03	.23 .03	.13 .03	.04 .02	.01 .01	1.55 .16	1.06 .10	.76 .07	.58 .05	.36 .02	.25 .02	1.80 .17	1.00 .09	.60 .06	.38 .04	.19 .02	.12 .01
	.86 .03	.42 .03	.23 .03	.10 .02	.05 .01		1.53 .05	1.12 .04	.85 .03	.51 .02	.32 .02		1.46 .10	.89 .05	.59 .04	.30 .02	.19 .01
	1.30 .04	.67 .03	.39 .03	.14 .02	.08 .02		2.07 .05	1.55 .04	1.2 .03	.75 .02	.52 .02		2.04 .09	1.32 .06	.88 .04	.47 .02	.29 .02
	2.45 .04	1.41 .03	.86 .03	.34 .02	.18 .01		3.28 .04	2.5 .01	2.05 .01	1.35 .01	.94 .01		3.26 .07	2.20 .06	1.62 .05	.87 .02	.54 .01

The top number in each compartment is U_N and bottom one V_N