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DEVIATIONS FROM ONE-DIMENSIONAL HEAT FLOW IN GUARDED HOT-PLATE MEASUREMENTS

BY

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Deviations from One-Dimensional Heat Flow in Guarded Hot-Plate Measurements

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A theoretical analysis is made of the error in thermal conductivity, measured by the guarded hot-plate apparatus, resulting from a temperature difference, or unbalance, between the test area and guard ring. The solution is obtained by the application of two successive Schwarz transformations, the assumptions having been verified by relaxation calculations. An expression is derived for the error heat flow in the test specimens. The agreement between calculated and measured values of the error heat flow for three different hot plates for which experimental data are available is 5% or better. It is therefore now possible to calculate the maximum tolerable unbalance to achieve any desired accuracy in thermal conductivity measured by the guarded hot plate.

THE standard apparatus for the determination of the thermal conductivity of building and insulating materials is the guarded hot plate.[1] The essential parts of the apparatus are shown schematically in Fig. 1. Two identical test specimens of known thickness are inserted between the plates and the assembly clamped together to ensure good thermal contact between adjacent surfaces. Constant temperature liquid is circulated through each of the cold plates and regulated current passed through the heaters of the test area and guard ring of the hot plate, so that a temperature gradient is set up across each specimen. When a steady-state temperature equilibrium has been established, the thermal conductivity of the test specimens may be calculated from the power input to the test area, the area of the test area, the temperature difference between the hot and cold sufraces of the specimens, and the thickness of the specimens. The formula by which the conductivity is calculated is derived assuming that one-dimensional normal heat flow occurs between the test area and cold plates.

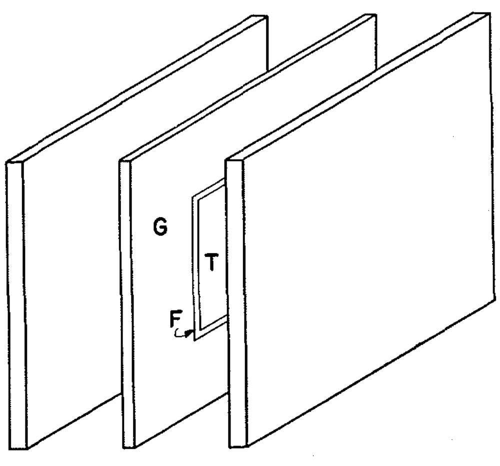

FIG. 1. Guarded hot plate apparatus. C, C—cold plates; S, S—positions of test specimens; H—hot plate; G—guard ring; T—test area; F—gap.

Unidirectional heat flow does not occur in the test region when (i) heat losses or gains at the edges of the specimens are excessive and/or (ii) a temperature difference exists between test area and guard ring. Errors due to (i) have been discussed by Somers and Cyphers,[2] Dusinberre,[3] and Woodside.[4]

To eliminate errors due to temperature differences between the test area and guard ring, the guard ring must be maintained at the same temperature as the test area so that no net heat exchange occurs between the two. This is done by varying, either manually or automatically, the magnitude of the guard ring heating current until the differential thermocouples connected across the gap separating test and guard areas register zero emf. When this is the case, the hot plate is said to be balanced. In practice, however, perfect balance is seldom achieved throughout the entire equilibrium period of a test. It is therefore important to be able to estimate the magnitude of the error in measured conductivity caused by small differences in temperature between the guard ring and test area.

Figure 2 represents a cross section of the apparatus through the center of the test area. Suppose the temperatures are as shown in the figure, the guard ring being 1°F cooler than the test area (1°F unbalance). A greater heat input to the test area will be required under these conditions than if the test and guard areas had both been at the same temperature of 96°F. If Q represents the rate of heat input to the test area under balanced conditions, and $(Q+q)$ the rate of heat input under the conditions shown in Fig. 2, then the relative error $\Delta k/k$ in the measured thermal conductivity is equal to q/Q. The error heat flow rate for a 1°F unbalance will be called q. It is composed of two parts

[1] American Society for Testing Materials, Designation C 177-45, "Standard Method of Test for Thermal Conductivity of Materials by Means of the Guarded Hot Plate," ASTM Standards, 1955, Part 3, pp. 1084–1092.

[2] E. V. Somers and J. A. Cyphers, Rev. Sci. Instr. 22, 583 (1951).

[3] G. M. Dusinberre, Rev. Sci. Instr. 23, 649 (1952).

[4] W. Woodside, "Analysis of Errors due to Edge Heat Loss in Guarded Hot Plates," Presented at ASTM Symposium on Thermal Conductivity Measurements, Philadelphia, Febraury 6, 1957 (to be published).

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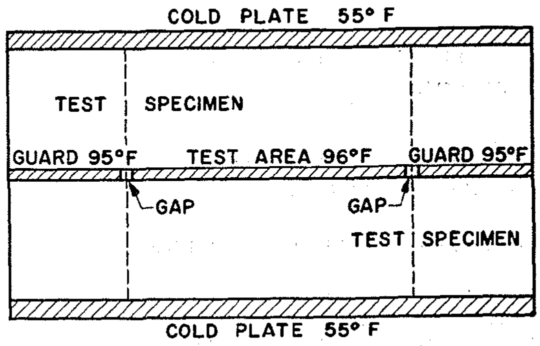

FIG. 2. Cross section of guarded hot-plate apparatus and test specimens through center of test area (not to scale).

(a) the heat transfer directly across the gap separating test area and guard ring q_0 and (b) the error heat flow rate in the test specimens q_1

$$\text{i.e., } q=q_0+q_1.$$

The heat flow between the test section of plate and specimens and the guard section of plate and specimens when there is an unbalance (i.e., heat flow across the dotted lines in Fig. 2) is the lateral heat flow rate q', and this also is composed of two parts (a) the heat transfer directly across the gap q_0 and (b) the lateral heat flow in the specimens q_1',

$$q'=q_0+q_1'.$$

The lateral heat flow q' is not the same as the error heat flow q. However it is the error heat flow q that determines the error in measured conductivity.

In a previous paper[5] which described experimental investigations of unbalance errors with three different hot plates, it was shown that

$$q=q_0+c\cdot k; \quad \text{i.e., } q_1=c\cdot k,$$

where q_0 and c are the unbalance sensitivity constants for any given hot plate. Values of q_0 and c were obtained for the three hot plates tested.

If the values of q_0 and c for a given hot plate are known, it is possible to calculate beforehand the maximum tolerable unbalance in any test which will keep the error in measured conductivity below any desired limiting value. Thus q_0 and c are important constants for any hot plate. It was shown in[5] how these constants may be determined experimentally and also how the value of q_0 may be calculated. However in the design of guarded hot plate apparatus it would be useful to be able to calculate the values of both constants for any particular heater plate before construction of the plate. This present paper is concerned with the analytical evaluation of the error heat flow in the test specimens $q_1=c\cdot k$ (and hence of the constant c) and also of the lateral heat flow in the specimens q_1'.

[5] W. Woodside and A. G. Wilson, "Unbalance Errors in Guarded Hot Plate Measurements," Presented at ASTM Symposium on Thermal Conductivity Measurements, Philadelphia, February 6, 1957 (to be published).

The analysis is based on assumptions that have been verified by relaxation calculations. The values of c predicted by the analysis are compared with the experimental values obtained for the three hot plates tested previously.[5]

ANALYSIS

Application of the Schwarz Transformation

For the purpose of evaluating the error heat flow and lateral heat flow in the test specimens due to a 1°F unbalance, the cross section shown in Fig. 2 may be reduced to that shown in Fig. 3. This is permissible for the following reasons: (i) since only the error and lateral heat flows in the specimens are of concern here (the heat flow q_0 directly across the gap may be evaluated independently) the heater plate of finite thickness may be replaced by a mathematical line; (ii) since only temperature differences are needed in heat conduction calculations, the temperatures shown in Fig. 2 may all be reduced by any equal amount, in this case 55°F. Also since the system is symmetrical about the two dotted lines shown, only the top left-hand corner of Fig. 3 need be considered.

The problem of evaluating the lateral heat flow across the dashed lines in Fig. 3 and the error heat flow out of the test area is still not amenable to mathematical solution.

The following assumptions are made. (a) The distortion in the isotherms and heat flow lines near the gap due to the unbalance does not extend to the edges of the specimen or to the center of the test area. This is a reasonable assumption since the ratio of temperature unbalance to hot-plate—cold-plate temperature difference (in the above case 1/41) is usually small, and the gap width is usually small compared with the linear dimension of the test area. This assumption was verified by a relaxation calculation of the temperature distribution in the test specimens for two designs of heater plate (Plates A and B of reference 5). Thus the specimens and plates may be considered to extend to infinity on each side of the gap for the purposes of lateral or error heat flow calculation.

(b) The temperature distribution in the test speci-

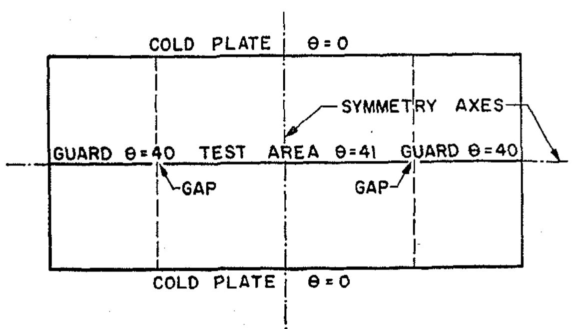

FIG. 3. Cross section of apparatus and test specimens, with hot and cold plates replaced by isotherms.

mens is the same as that which would be obtained considering only two-dimensional heat flow. The error involved in this assumption has also been shown to be very small by relaxation calculations. Thus the system may be handled by the application of a two-dimensional conformal transformation. Also since the differential equation for heat conduction is linear, the lateral and error heat flows in the specimens, under the temperature conditions shown in Fig. 3, are the same as under the following temperature conditions: test area temperature $\theta=1$, guard ring temperature $\theta=0$, cold plate temperature $\theta=0$. The system has therefore been simplified to that shown in Fig. 4.

Choose C to be the origin of an x-y Cartesian coordinate system. Let the gap width be $2d$, the linear dimension of the test area $2(l+d)$ (so that B represents the center of the test area) and the thickness of the specimen h. The temperature boundary conditions are as shown in the figure. The boundary condition at the gap requires zero heat flow normal to the gap width, and therefore along the gap, $\partial\theta/\partial y=0$.

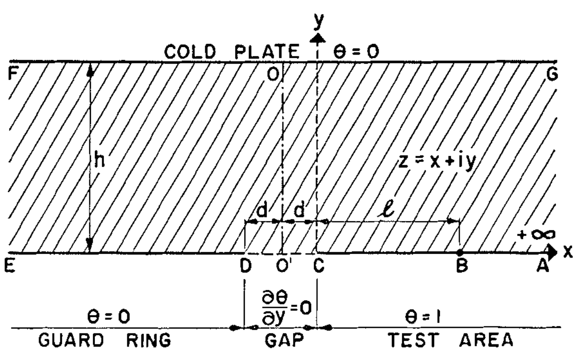

FIG. 4. Simplified plate-specimen heat flow system showing reduced temperature boundary conditions.

The determination of the lateral heat flow across the dashed line bisecting the gap, or of the heat flow out of the test area CB, is not simple in the system shown. Figure 4 will therefore be transformed to a simpler system in which the solution is obvious, by means of two successive Schwarz-Christoffel transformations. Figure 4 will be taken to represent the system in the complex $z=(x+iy)$ plane, and, as a first step, will be transformed into the $w=(u+iv)$ plane shown in Fig. 5. The space inside the parallel lines in Fig. 4 is transformed to the space above the u axis in Fig. 5 with the points A to G in correspondence, by the following Schwarz transformation:

$$dz/dw=b/(w+a).$$

Integrating,

$$z=b[\ln(w+a)+b'],$$

where b and b' are constants yet to be determined. When $z=0$, $w=0$, since C is the origin in both planes.

$$\therefore b'=-\ln a$$

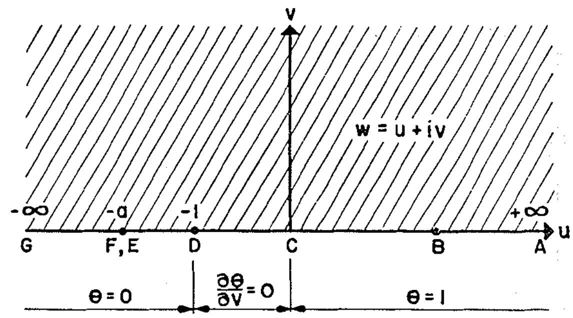

FIG. 5. Representation of Fig. 4 after application of first transformation.

and

$$z=b\ln[(w/a)+1].$$

Along FG, w is real and less than $-a$, and the imaginary part of z is ih,

$$\therefore b=h/\pi.$$

Setting $w=-1$ when $z=-2d$, then

$$a^{-1}=1-e^{-2\pi d/h}. \tag{1}$$

Thus the equation transforming Fig. 5 to Fig. 4 is

$$z=(h/\pi)\ln[(w/a)+1]. \tag{2}$$

Figure 5 will now be transformed to the $t=(r+is)$ plane shown in Fig. 6, by a second Schwarz transformation:

$$dt/dw=b_1[w(w+1)]^{-\frac{1}{2}}.$$

Integrating,

$$t=ib_1\sin^{-1}(2w+1)+b_1',$$

where b_1 and b_1' are constants. When $t=0$, $w=0$; $\therefore b_1'=-\frac{1}{2}i\pi b_1$. When $t=-\pi$, $w=-1$; $\therefore b_1=-i$.

$$\therefore t=\sin^{-1}(2w+1)-\tfrac{1}{2}\pi. \tag{3}$$

Equation (3) therefore transforms Fig. 5 into Fig. 6. The single transformation which transforms Fig. 4, the

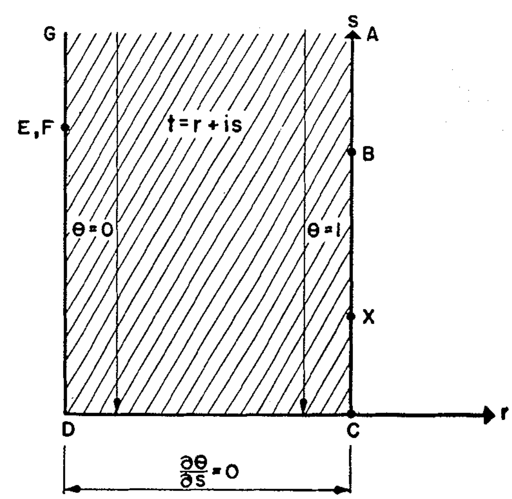

FIG. 6. Representation of Fig. 4 after application of two successive transformations.

original system, directly into Fig. 6, is obtained by eliminating w from Eqs. (2) and (3). This results in

$$\cos t = 2a(e^{\pi z/h}-1)+1. \tag{4}$$

The solution of the problem in the $t=(r+is)$ plane of Fig. 6 is straightforward since ABC is at temperature $\theta=1$, $DEFG$ at temperature $\theta=0$, there is no heat flow across CD, (i.e., $\partial\theta/\partial s=0$) and A and G are at $s=+\infty$. The isotherms are therefore given by $r=$constant and the heat flow lines by $s=$constant, and the temperature distribution is given by

$$\theta=(r/\pi)+1. \tag{5}$$

Evaluation of the Lateral Heat Flow

As a first solution, the lateral heat flow crossing the line 00′ in Fig. 4 will be determined. The line 00′ bisects the gap between test area and guard sections of the specimen. In Fig. 6 the point corresponding to 0′ in Fig. 4 is somewhere on DC, the exact location being unimportant, since in Fig. 6, the heat flow is one-dimensional. However the ordinate (value of s) for point 0 in the t plane must be determined.

At 0 in the z plane, $z=(-d+ih)$. Substituting this into Eq. (4),

$$\cos t = -2a(e^{-\pi d/h}+1)+1.$$

But $\cos t=\cos(r+is)$ and since $r=-\pi$ for point 0, $\cos t=-\cosh s$. Therefore, at 0,

$$s=s_0=\cosh^{-1}\left(\frac{e^{\pi d/h}+1}{e^{\pi d/h}-1}\right)$$

$$=\cosh^{-1}[\coth \pi d/2h].$$

Therefore the heat flow across the line 00′, for unit depth perpendicular to the plane of the figure, is equal to

$$k\cdot\frac{s_0}{144}\cdot\frac{1}{\pi}.$$

Hence for a heater plate with a test area, perimeter of $8(l+d)$, the lateral heat flow in both specimens for a 1°F unbalance, neglecting corner effects, is given by

$$q'-q_0=2k\cdot\frac{s_0}{144}\cdot 8(l+d)\cdot\frac{1}{\pi}.$$

Substituting for s_0,

$$q'-q_0=\frac{k(l+d)}{9\pi}\cosh^{-1}[\coth \pi d/2h]$$

$$\text{Btu/hr °F unbalance.} \tag{6}$$

Evaluation of the Error Heat Flow

The above solution for the lateral heat flow assumed a two-dimensional temperature distribution in the region of the test specimen adjacent to the gap, and the heat flow crossing 00′ was evaluated neglecting the fact that the heater plate test area is square, and therefore the heat flow crossing from test section to guard section of the specimen is three-dimensional. The following solution for the error heat flow retains the assumption that the two-dimensional temperature distribution is close enough to the actual distribution as to have negligible effect on the heat flow out of the test area, but the heat flow out of the test area will be calculated taking account of the three-dimensional nature of the flow.

In the z plane (Fig. 4) point B corresponds to the center of the test area. Consider a point X distant x from C on the line BC. The point corresponding to X in the t-plane will be determined. Since X is on BC, the value of r at X is zero. From Eq. (4) the value of s at X is given by

$$s_x=\cosh^{-1}(2a(e^{\pi x/h}-1)+1).$$

For the heat flow out of XC, for unit depth perpendicular to the plane of Fig. 6, in the t plane, one obtains

$$k\cdot\frac{s_x}{144}\cdot\frac{1}{\pi}.$$

But the heat flow out of XC, for unit depth perpendicular to the plane of Fig. 4, in the z plane is

$$-\frac{k}{144}\int_0^x\left(\frac{\partial\theta}{\partial y}\right)_{y=0}dx.$$

Equating these two expressions for the heat flow out of XC, and differentiating both sides with respect to x,

$$-\left(\frac{\partial\theta}{\partial y}\right)_{y=0}=\frac{1}{\pi}\frac{d}{dx}\cosh^{-1}(2a(e^{\pi x/h}-1)+1).$$

Therefore the total heat flow out of the test area, taking account of the three-dimensional nature of the flow, and assuming that the isotherms are concentric squares centered on B, is

$$Q+(q-q_0)=-\frac{2k}{144}\int_0^l 8(l-x)\left(\frac{\partial\theta}{\partial y}\right)_{y=0}dx.$$

Expanding, and then integrating by parts, this becomes

$$Q+(q-q_0)=\frac{k}{9\pi}\int_0^l\cosh^{-1}(2a(e^{\pi x/h}-1)+1)dx.$$

Let this integral be denoted by I, since it can only be evaluated numerically, then

$$Q+(q-q_0)=kI/9\pi.$$

But Q, the total heat flow from both surfaces of the test area, under balanced conditions when both test area and guard ring are at temperature $\theta=1$, is given by

$$Q=k(l+d)^2/18h.$$

Therefore the error heat flow in the specimens $(q-q_0)$ for a 1°F unbalance is

$$q-q_0=(kI/9\pi)-k(l+d)^2/18h.$$

Therefore

$$\left.\begin{aligned} c&=(q-q_0)/k=I/9\pi-(l+d)^2/18h \\ &\text{where} \\ I&=\int_0^l \cosh^{-1}(2a(e^{\pi x/h}-1)+1)dx \\ &\text{and} \\ a^{-1}&=1-e^{-2\pi d/h} \end{aligned}\right\}. \quad (7)$$

The evaluation of the integral I defined in Eq. (7) requires the plotting of the function

$$y=\cosh^{-1}(2a(e^{\pi x/h}-1)+1)$$

and the determination of the area under this curve between the limits $x=0$ and $x=l$, by Simpson's rule or the use of planimeter. An approximate value of I which is more easily evaluated is given by

$$I\simeq(\pi l^2/2h)+l\cdot\ln(4a)$$

and hence the approximate value of c corresponding to this value of I is

$$c\simeq(l/9\pi)\ln(4a)$$

or

$$c\simeq0.0814l\cdot\log_{10}(4a), \quad (8)$$

where a has the same value as before.

The parameter c is not strictly a constant for a given hot plate, its value depending upon the thickness of the specimens being tested. Table I shows values of c

TABLE I. Effect of specimen thickness upon the error heat flow for hot plate with $2(l+d)=4$, and $2d=0.0625$ in.

Specimen thickness h, (in.)	Calculated value of c, Eq. (8)
0.25	0.141
0.50	0.177
1.00	0.220
2.00	0.265

calculated from the approximate Eq. (8) for different specimen thicknesses for a hot plate having $2l=3\frac{15}{16}$ in. and $2d=\frac{1}{16}$ in. (the dimensions of the 8-in. National Bureau of Standards design).

COMPARISON WITH EXPERIMENTAL RESULTS

Equations (6), (7), and (8) are now applied to the calculation of the lateral heat flow, the error heat flow,

TABLE II. Comparsion of calculated and experimental results.

Dimensions of hot plate and specimens (in.)				Error or lateral heat flow in specimens (Btu/hr °F unbalance)			
Size of hot plate	Side of test area $2(l+d)$	Gap width $2d$	Specimen thickness h	Calculated equation (6)	Calculated equation (8)	Calculated equation (7)	Measured error heat flow
8×8	4	0.0625	1.0	$0.262k$	$0.220k$	$0.203k$	$0.192k$
8×8	4	0.1250	1.0	$0.213k$	$0.177k$	$0.164k$	$0.170k$
18×18	12	0.0935	2.0	$0.849k$	$0.717k$	$0.679k$	$0.680k$

and the approximate error heat flow, respectively, in the test specimens, for heater plates A, B, and C of (5) for which experimental values of the error heat flow in the specimens are available. The comparison between calculated and experimental values is shown in Table II.

In all cases the lateral heat flow is larger than the error flow. The agreement between the calculated and experimental values of the error heat flow is satisfactory, being 5% or better. The approximate error heat flow calculated from the simpler equation (8) differs by not more than 8% from the accurate error heat flow [Eq. (7)] and therefore should be suitable for design calculations.

NOMENCLATURE

- Q = rate of heat input to test area,
- q = total error heat flow for a 1°F unbalance, Btu/hr °F unbalance,
- k = thermal conductivity of test specimens, Btu in./hr sq ft °F,
- Δk = error in thermal conductivity due to 1°F unbalance, Btu in./hr sq ft °F,
- q_0 = heat transfer directly across gap, Btu/hr °F unbalance,
- q_1 = error heat flow rate in test specimen, Btu/hr °F unbalance,
- q' = total lateral heat flow, Btu/hr °F unbalance,
- q_1' = lateral heat flow rate in test specimens, Btu/hr °F unbalance,
- θ = temperature,
- c = slope of q *vs* k graph,
- h = thickness of test specimen, in.,
- $2d$ = width of gap separating test area and guard ring, in.,
- $2l$ = linear dimension of test area plate, in.

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