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Combined stochastic and transfer model for atmospheric radiation

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Abstract

A model for incident solar radiation is developed. The sky is treated as if it were composed of two semi-grey layers, an upper passively attenuating medium, and a lower participating isotropically scattering medium. The radiative transfer equation is solved for the lower medium using an exponential kernel approximation. This solution is combined with a gamma distribution for the creation of cloud particles to obtain a probability distribution for optical thickness. The combined solution is fitted and compared to generalised radiation curves.

Keywords: Radiative transfer; Atmospheric radiation; Solar energy; Probability density function; Modelling

1. Introduction

The nature of solar radiation has been of interest for many years. Ångström [1] was among the first to publish a regression analysis designed to predict local insolation, based on the sunshine fraction, n/N, where n are the number of hours of bright sunshine and N are the total number of daylight hours. Hottel and Whillier [2] considered the observed cumulative frequency distribution, $F(H/\bar{H})$, as a function of the ratio of insolation on a horizontal surface, H, to the long-term average value, $\bar{H}$. Liu and Jordan [3] published data for $F(k_T)$, where k_T is a clearness index,

$$k_T = \frac{H}{H_0} \tag{1}$$

i.e., the ratio of incident energy to extraterrestrial energy on a horizontal surface, H_0. An important observation made by the authors was that the form of the cumulative frequency distribution, $F(k_T)$, for horizontal surfaces, appeared to be essentially universal for a given value of the mean clearness

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index, $\bar{k}_T$. These so-called, 'generalised radiation' distribution curves attracted much interest at the time. Although subsequent research [4,5] suggested the frequency of insolation to be a function of local weather conditions, time-of-year etc., the authors' contribution is still useful due to the reduction in parameters resulting from a single set of 'universal' cumulative frequency distributions.

Substantial additional research into atmospheric radiation are summarised in the reports of the International Energy Agency, Solar Heating and Cooling Program [6,7]. Models are categorised as cloudless-sky, sunshine, total cloud, cloud-layer and 'Liu and Jordan' schemes.

The simplest cloudless-sky models are based on Beer's law with a sky-beam transmittance, k_{sb}, given by

$$k_{sb} = \frac{G_{sb}}{G_0} = \exp(-\tau_L/\mu_0), \tag{2}$$

where G_{sb} is sky-beam irradiation and $G_0 = \mu_0 G_{sc}$ is horizontal extraterrestrial irradiation, on a horizontal surface; μ_0 is the cosine of the zenith angle, $\theta_0 = \cos^{-1}(\mu_0)$, and G_{sc} is the solar constant having a nominal value of 1363 W/m^2. In general the optical thickness, τ_L, contains terms due to both absorption and scattering

$$\tau_L = \int_0^L \beta\,\mathrm{d}x = \int_0^L (\kappa + \sigma_s)\,\mathrm{d}x \tag{3}$$

with an effective transmittance being computed for absorption by greenhouse gases; water vapour, carbon dioxide and also molecular (Rayleigh) and aerosol (Mie) scattering. Various other formulations such as that of Hottel [8], have been proposed. Although technically, Beer's law is based on the premise of a negligibly scattering cold medium, some cloudless sky models also presume a fraction of the (in-scattered) beam radiation is transmitted downwards as clear-sky diffuse radiation, G_{sd}, with a 'pseudo-transmittance', $k_{sd} = G_{sd}/G_0$. For molecular scattering the downward and upward scattering components are often chosen as equal, while for aerosols it is well known that the scattering is anisotropic. The scattering albedo, together with the order of the layers are sufficient to compute G_{sd}.

Sunshine, total-cloud, and cloud-layer based models may or may not incorporate a cloudless sky model. Davies et al. [7] propose that the prototype for these be a modified Ångström equation,

$$\frac{G}{G_s} = \frac{k + (1-k)n/N}{1 - \rho_g \rho_c}, \tag{4}$$

where k is a transmittance, and ρ_g and ρ_c are the ground and cloud reflectances. The denominator differs from unity owing to back-reflection of ground radiosity by clouds. Total clear-sky radiation on a horizontal surface $G_s = G_{sb} + G_{sd}$ is sometimes replaced by the horizontal extraterrestrial radiation, G_0. For total-cloud based models, the sunshine-fraction is replaced by the total cloud amount, $c = 1 - n/N$, while for cloud-layer based models, cloud amounts c_i, with transmittances, k_i, for different cloud types are summed so the numerator is $\sum_{i=1}^{n}(1-c_i)+k_i c_i$. 'Liu and Jordan' models refer to the fact that, in addition to proposing the notion of 'generalised radiation', the authors also suggested the diffuse-to-total radiation, G_d/G_0, to be principally a function of k_T. However, Ruth and Chant [9] observed the G_d/G_0 profile to be a function of latitude, and Bugler [10] and Iqbal [11] noted variations in G_d/G_0 as a function of altitude angle.

The report by Davies et al. [7] contains a comprehensive evaluation of the five model types described above, with experimental data for Europe, North America, and Australia. The authors concluded that direct-beam radiation is best predicted by cloud-layer based models, followed by total-cloud and sunshine models, and that 'Liu and Jordan' methods predicted diffuse radiation most accurately. The majority of the existing models are based on simple numerical correlations fitted to observed physical data, and contain little or no heat transfer theory (Hollands [12] is a notable exception). Since such models cannot, be considered predictive tools, it was decided to generate a new formulation, based on sound mathematical principles.

2. Present study

A two-layer model, notionally similar to that suggested in Ref. [12] and illustrated in Fig. 1, combining both statistics and heat transfer is proposed. A cloudless-sky model may be applied to the upper passive medium, for which the optical thickness is presumed constant. The particular choice of clear-sky model is not prescribed in this paper, since attention is primarily directed to the lower medium. The latter is presumed to attenuate radiation by scattering, and an approximate solution to the radiative transfer equation (RTE) is obtained; it being assumed that the optical thickness varies due to cloud action. The model is then compared with the Liu and Jordan cumulative frequency distribution, and used to explore circumstances under which it reasonable to presume that the notion of 'generalised radiation' is valid.

2.1. Radiative transfer

Let it be assumed that the lower medium is a grey isotropically-scattering medium in radiative equilibrium, bounded by a source of diffuse radiation at the ground below, and sources of beam and

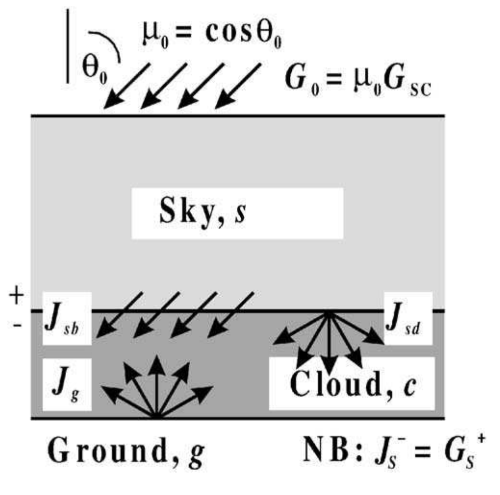

Fig. 1. Schematic of radiation model.

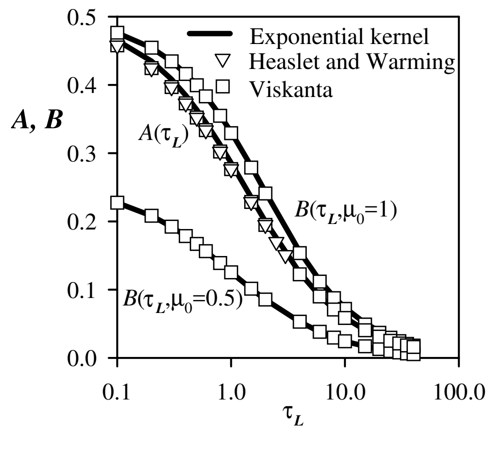

Fig. 2. Values of the coefficients A and B, compared to those of Viskanta [13] and Heaslet and Warming [17] and expressed as a function of optical depth, τ_L, in the normal direction.

diffuse radiation from the clear sky above, so that the boundary conditions are,

$$I^+(0,\mu) = \frac{J_g}{\pi}, \tag{5}$$

$$I^-(\tau_L,\mu) = \delta(\mu-\mu_0)\frac{J_{sb}}{\mu_0} + \frac{J_{sd}}{\mu_0}, \tag{6}$$

where I is radiation intensity, J_{sb} and J_{sd} are beam and diffuse components of the sky radiosity, J_g is the ground radiosity, $\mu = \cos\theta$, and δ is the Dirac delta function. Under the circumstances, it can be shown that the net radiation flux, $\dot{q}$, is

$$\dot{q} = 2\left(J_g E_3(\tau_L) - \frac{1}{2}\frac{J_{sb}}{\mu_0}\exp-(\tau_L-\tau)/\mu_0 - J_{sd}E_3(\tau_L-\tau)\right) + 2\left(\int_0^{\tau} S(\tau^*)E_2(\tau-\tau^*)\,d\tau^* - \int_{\tau}^{\tau_L} S(\tau^*)E_2(\tau-\tau^*)\,d\tau^*\right) \tag{7}$$

with

$$S(\tau) = \frac{1}{2}\left(J_g E_2(\tau) + \frac{1}{2}\frac{J_{sb}}{\mu_0}\exp-(\tau_L-\tau)/\mu_0 + J_{sd}E_2(\tau) + \int_0^{\tau_L} S(\tau)E_1|\tau-\tau^*|\,d\tau^*\right) \tag{8}$$

where E_n are exponential integrals defined in the Appendix. Viskanta [13] derived a solution for the radiative flux at the ground, namely,

$$\dot{q}_g = J_g - G_g = 2\left(A(J_g - J_{sd}) - B\frac{J_{sb}}{\mu_0}\right), \tag{9}$$

where G_g is the total irradiation on a horizontal ground surface. Viskanta computed values of $A(\tau_L)$ and $B(\tau_L,\mu_0)$ using an approximate collocation scheme. Here, algebraic expressions for A and B are required, and an exponential kernel method yields,

$$A = \frac{2}{4+3\tau_L}, \tag{10}$$

$$B = \mu_0\frac{(2+3\mu_0)+(2-3\mu_0)\exp(-\tau_L/\mu_0)}{2(4+3\tau_L)}. \tag{11}$$

Eq. (10) is the well-known solution for a one dimensional (1-D) layer with diffuse boundaries which may be found in any text on radiation heat transfer [14,15], while Eq. (11) is similar to the expression derived in detail in Buckius and King [16], but based on the kernel approximations $E_2(\tau_L) \cong \frac{3}{4}\exp(-\frac{3}{2}\tau_L)$, $E_3(\tau_L) \cong \frac{1}{2}\exp(-\frac{3}{2}\tau_L)$ consistent with Eq. (10) as derived in Refs. [14,15]. Fig. 2 is a comparison with Refs. [13,17]. Values of B are in excellent agreement, while values for A are slightly higher than [13,17].

The ground radiosity is given by,

$$J_g = \varepsilon_g E_b + \rho_g G_g, \tag{12}$$

where ε is the ground emittance, E_b is black body emissive power, and ρ_g, is ground reflectance. It is further assumed that earth's emittance is entirely in the infra-red spectrum, so the solar component

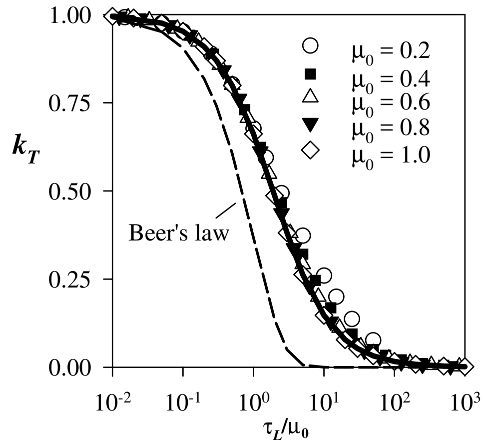

Fig. 3. Clearness index k_{T} vs. optical depth in the zenith direction, τ_{L}/μ_0, for the case $\rho_{\mathrm{g}} = 0$, $k_{\mathrm{sb}} = 1$, $k_{\mathrm{sd}} = 0$.

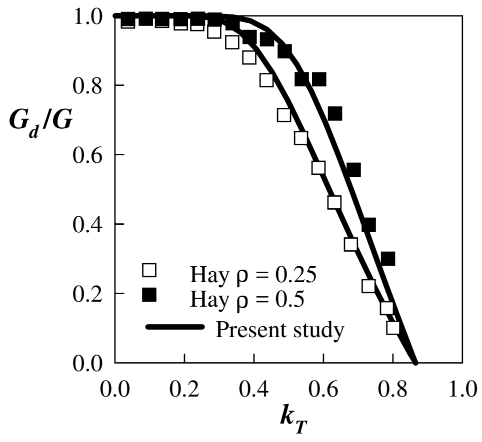

Fig. 4. Diffuse fraction vs. clearness index compared with Hay's data [18] for $\rho_{\mathrm{g}} = 0.25$ and 0.5, with $k_{\mathrm{sb}} = 0.865$, $k_{\mathrm{sd}} = 0$, $\mu_0 = 0.5$. NB: Under the circumstances $k_{\mathrm{T}} = 2B/\mu_0$.

of the radiosity is entirely due to reflection only. Under the circumstances the first term on the right-side of Eq. (12) is neglected and the clearness index, k_{T}, given by,

$$k_{\mathrm{T}} = 2\frac{Ak_{\mathrm{sd}} + Bk_{\mathrm{sb}}/\mu_0}{[1 - \rho_{\mathrm{g}}(1 - 2A)]}. \tag{13}$$

Eq. (13) differs from Eq. (4) in that it suggests k_{T} to be a function of μ_0, and also of k_{sd} and k_{sb}. However inspection of Fig. 3 shows that when k_{T} is plotted as a function of τ_{L}/μ_0 for the simplified case $\rho_{\mathrm{g}} = 0$, $k_{\mathrm{sb}} = 1$, $k_{\mathrm{sd}} = 0$, there is little variation as a function of μ_0: For the notion of generalised radiation to have validity, it is necessary to show directional (air mass) effects are small. This figure shows that the term $2B/\mu_0$ is effectively a function of τ_{L}/μ_0, with only minor departures for low μ_0 and high τ_{L} due to the diffuse component of the attenuated beam radiation. The solid line in Fig. 3 is based on the correlation

$$B = \frac{3\mu_0}{6 + 3\tau_{\mathrm{L}}/\mu_0}, \tag{14}$$

which is a purely numerical fit and which gives a better fit to the data than simply substituting $B/\mu_0 = A(\tau_{\mathrm{L}}/\mu_0) = 2/(4 + 3\tau_{\mathrm{L}}/\mu_0)$ from Eq. (10), which nonetheless still gives a much better fit to the data than Beer's law. The difference between the two solutions is just that between the solution for pure scattering, and that for a cold absorbing medium, where the emitted or in-scattered component of radiation is excluded.

Fig. 4 shows the ratio of diffuse-to-total irradiation

$$\frac{G_{\mathrm{d}}}{G} = 1 - \frac{k_{\mathrm{sb}}\exp(-\tau_{\mathrm{L}}/\mu_0)}{k_{\mathrm{T}}} \tag{15}$$

on a horizontal surface compared with Hay's data for $\rho_{\mathrm{g}} = 0.25$ and 0.5. These results compare favourably with the data, and also with Hollands heat transfer model [12,18]. The dependence of the diffuse fraction upon ground reflectance cannot be predicted using a conventional Liu and Jordan diffuse-to-total radiation model.

2.2. Random events

Let it be assumed that scattering cloud particles occur randomly in time. Bendt et al. [19] considered an exponential distribution for $p(k_{\mathrm{T}})$, whereas Feuillard et al. [20] presumed a Poisson distribution, and derived a relationship for $p(k_{\mathrm{T}})$ based on a sunshine-type model. Here, a similar approach to [20] is followed, but including the RTE solution: If the probability of a particle at a point over time Δt is a', then the probability density function for b particles occurring in time t is a Poisson distribution. The probability density function of the time between particle generations is a gamma distribution. Let it be assumed that the local extinction coefficient, β, is proportional to the number b, and inversely proportional to the time t, i.e. $\beta = \alpha b/t$, where α is just a constant of proportionality. Furthermore let it be assumed that the cloud layer is homogeneous so that $\tau_{\mathrm{L}} = \beta L$. Under the circumstances

$$p(\tau_{\mathrm{L}}) = \frac{(a/\tau_{\mathrm{L}})^{\mathrm{b}} \exp(-a/\tau_{\mathrm{L}})}{\tau_{\mathrm{L}} \Gamma(b)}, \tag{16}$$

where $a = \alpha a' bL$ is referred to as a rate parameter, b is a shape parameter, and Γ is a gamma function, defined in the Appendix. The probability density function, $p(k_{\mathrm{T}})$, is computed from $k_{\mathrm{T}}(\tau_{\mathrm{L}})$ using

$$p(k_{\mathrm{T}}) = \frac{p(\tau_{\mathrm{L}})}{|\partial k_{\mathrm{T}}/\partial \tau_{\mathrm{L}}|} \tag{17}$$

and the cumulative probability distribution may similarly be written as,

$$P(k_{\mathrm{T}}) = \frac{\gamma(a/\tau_{\mathrm{L}}, b)}{\Gamma(b)}, \tag{18}$$

where γ is an incomplete gamma function (see the Appendix). This constitutes the mathematical description of the model outlined in this paper.

2.3. Comparison with Liu and Jordan data

Trial calculations showed b to be close to unity almost invariably, when a was chosen so the mean value of $\bar{k}_{\mathrm{T}}$ is correct. Fig. 5 shows excellent agreement for $P(k_{\mathrm{T}})$ with $b = 1$, for $\mu_0 = 1$, $k_{\mathrm{sb}} = 0.92$, $k_{\mathrm{sd}} = \rho_{\mathrm{g}} = 0$, compared to the data of Liu and Jordan [3], for $0.3 \leqslant \bar{k}_{\mathrm{T}} \leqslant 0.6$. For $\bar{k}_{\mathrm{T}} = 0.7$ however, good agreement could only be obtained by adjusting b to a value of 3. Fig. 6 shows the probability density function, $p(k_{\mathrm{T}})$, for $\bar{k}_{\mathrm{T}} = 0.7$, $b = 1$. The global maximum of the probability function is nominally the same for $b = 1$ and $b = 3$, but occurring at a higher value of k_{T}, and with a slightly narrower profile. However the two profiles are otherwise quite similar. Table 1 gives the values of the rate and shape coefficients used to generate Figs. 5 and 6: The reader will note that these are based on values of B computed using Eq. (11) and not the simplified Eq. (14).

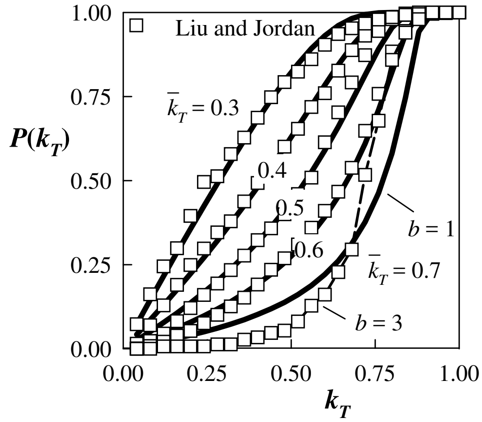

Fig. 5. Cummulative probability, $P(k_T)$ in the range $\bar{k}_T = 0.3$–0.7, compared with the experimental data of Liu and Jordan [3].

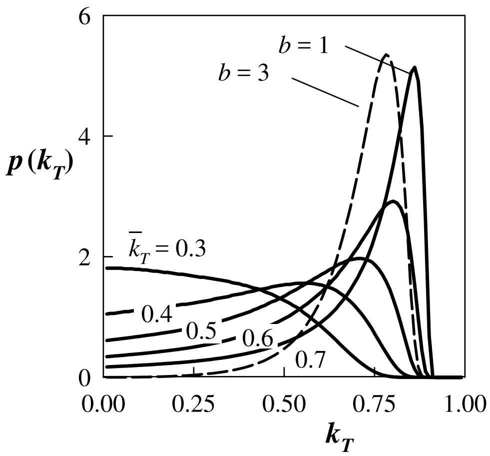

Fig. 6. Probability density function, $p(k_T)$ in the range $\bar{k}_T = 0.3$–0.7.

Table 1
Values of the rate and shape coefficients used to generate data for Figs. 5 and 6, $\rho = 0, k_{sb} = 0.92$

$\bar{k}_T$	a	b
0.3	2.785	1.0
0.4	1.598	1.0
0.5	0.928	1.0
0.6	0.519	1.0
0.7	0.260	1.0
0.7	1.282	3.0

3. Discussion of results

The results provide quantitatively accurate insight into the nature of solar radiation, with little or no calibration. With $b = 1$, $p(\tau) = a/\tau_L^2 \exp(-a/\tau_L)$ and $P(\tau_L) = \gamma(a/\tau_L, 1)$, and the distribution is solely dependent on the rate parameter, a. This is a positive trend for the existence of a universal profile, as postulated in Liu and Jordan [3], other than for high $\bar{k}_T$ where there appears to be a dependence on the shape parameter b. Even here, Fig. 6 suggests that for high $\bar{k}_T$, the $b = 1$ profile is qualitatively correct, only the peak value of $p(k_T)$ occurs at a higher value of k_T than for $b = 3$. One problem with Liu and Jordan's data [3] is that other effects, due to air mass, clear-sky extinction, and ground reflectance are present in the data, and need to be eliminated, prior to a true assessment being undertaken.

In deriving the data for Figs. 5 and 6, normal incidence, $\mu_0 = 1$, was presumed: The Liu and Jordan data were gathered over a wide range of air-mass (nominally $1/\mu_0$) conditions, yet this does not appear to have had an influence on the radiation curves. It has already been noted that the expression $2B/\mu_0$ in Eq. (13) may be considered primarily a function of τ_L/μ_0. Considering the probability density function $p(\tau_L/\mu_0)$, this is also a function only of τ_L/μ_0, i.e., τ_L/μ_0 is a parameter in expressions for both p and $2B/\mu_0$. The value $\bar{k}_T$ will decrease, since the rate constant is not a but a/μ_0, causing the profile to shift, but provided b is constant, the profile will still belong to the same family of curves. This may explain why measured $F(k_T)$ are a function of $\bar{k}_T$ alone, regardless of significant hourly changes in μ_0. The model equations thus provide some quantitative reinforcement as to the existence of a so-called 'universal' insolation distribution, at least for as long as b is constant in Eq. (16). Directional effects will also manifest themselves in Eq. (13) through the clear sky transmittances, k_{sb} and k_{sd}, and the ground radiosity term for the case $\rho_g > 0$. However if k_{sb} is close to unity and k_{sd} and ρ_g are small, these effects should be subordinate.

The influence of beam and diffuse transmittances upon the probability density function was also investigated. The effect of the beam component of the clear sky transmittance is to decrease the range of the abscissa from 1 to k_{sb}, i.e. decreasing k_{sb} causes the profiles to compress horizontally and increase in height. For low $\bar{k}_T$ profiles the model is relatively insensitive to the choice of k_{sb}, but not for high $\bar{k}_T$. The impact of the 'diffuse transmittance', k_{sd}, is to decrease values of $p(k_T)$ towards the origin for low k_T, i.e. to flatten out the profile; a trend consistent with observations for temperate storm zones [4]. Ground effects also appear to have a significant influence on $p(k_T)$ reducing the variance of the distribution with increasing ρ_g, and leading to higher global maxima in the probability density functions.

Comparison of this model with existing sunshine and cloud-based models reveals similarities and differences: (a) Here the cloud transmittance is accounted for by solving the RTE. (b) Mean insolation is obtained by integrating $k_T\,p(k_T)$ over a continuum of all possible values. For sunshine and cloud-based models, (a) the heat transfer analysis is based on lumping the parameters, eliminating variables, (b) a discrete probability distribution is summed, with relatively few states, e.g. two in Eq. (4) 'sunny' with $p=n/N$, $k_{sb}=1$ and 'cloudy' with $p=1-n/N$, $k_{sb}=k$. The generally-superior cloud-layer models [7] allow for a few more states corresponding to different cloud 'types'. Although the latter implies some allowance is made for cloud-layer interactions, they are ensemble-averaged as is the present model. Here, the terms $2A$ and $2B/\mu_0$ are diffuse-sky, and direct and attenuated beam incident radiation, while the denominator includes back-reflection, i.e., $(1-2A)=\rho_c$ in Eq. (13), based on the assumption of isotropic scattering [18].

In this model the existence of a uniform grey cloud-layer of constant thickness was proposed. There can be few places in the world where cloud coverage is of such a form: For any real situation there will be significant spatial variation in the extinction coefficient, β, so Eq. (13) does not entirely describe the heat transfer process, as cloudiness variations are ignored in the 1-D RTE. However, the 1-D solution to the RTE represents a reasonable idealization, which does not fundamentally compromise the stochastic component of the model. One might speculate that at very high $\bar{k}_T$, when clouds seldom occur, that they may manifest themselves as random, coherent, 3-D structures, and a layer model may be inappropriate in that context. Such questions might be explored further, by conducting Monte Carlo simulations, or other numerical analyses, based on presumed cloud type, pattern, shape, and opacity, for 'standard days' from observed meteorological conditions, and thence deriving values of A and B in Eq. (13), numerically.

4. Conclusion

A model for describing atmospheric radiation was described. There are two main components to the model: (1) a heat transfer model, and (2) a probability density function for cloudiness. The radiation model predicts the ratio of beam-to-diffuse radiation as a function of altitude and air-mass, unlike many conventional models, which are mostly empirical with little or no heat transfer theory. The statistical model presumes cloud particles occur randomly, and contained two parameters referred to as the rate and shape factors. By adjusting the former to obtain the correct value of $\bar{k}_T$, the model correlated well with the data of Liu and Jordan for constant shape factor, except for very high clearness. This may be due to air-mass-related or ground reflectance effects, or because the notion of generalised radiation, however meritorious, may nonetheless have finite limitations. The concept of generalised radiation is valid within the context of this model if: the term due to attenuated sky radiation is predominant in the radiation model; the influence of ground reflectance, clear sky beam and diffuse transmittances are subordinate, and the probability distribution is a function of one parameter, τ_L/μ_0, which determines the value of $\bar{k}_T$. The statistical model may also be used to describe local insolation for climates, which do not exhibit a generalised radiation distribution, based on the ability to adjust the shape and rate parameters to local conditions. There is of course no reason not to use other probability distributions or heat transfer models, should these prove superior.

5. Future work

There are a number of ways in which the present model may be extended, for example, ozone and water vapour fractions were presumed constant; these too could be considered to be stochastic processes, and Eq. (16) applied. Only two media were considered here (though multiple layers are tacitly implied in the upper medium): a multi-layer model could readily be constructed with known probability density functions and radiative properties for each medium; O_3, H_2O, CO_2, aerosols, and various cloud types, at different heights with a distinction being made between absorbing and scattering layers. The requirement for isotropy may also be relaxed. Radiation network methods [21] would allow for the solution for such a system with a minimum of iteration. In this latter situation, Eq. (13) could be applied on a spectral basis, and integrated over wavelength. Fitting the model to more extensive field-data would be of use in evaluating the merits of such modifications in the future.

Appendix

The exponential integral, Abramowitz and Stegun [22], is defined according to,

$$E_n(x) = \int_1^\infty \frac{\exp(xt)}{t^n}\,\mathrm{d}t. \tag{A.1}$$

The gamma function is defined according to

$$\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)\,\mathrm{d}t. \tag{A.2}$$

This was evaluated according to Hastings [22] as,

$$\Gamma(x+1) = \sum_{i=1}^{8} b_i x^i, \tag{A.3}$$

where $b_1=-0.577191652$, $b_2=0.988205891$, $b_3=-0.897056937$, $b_4=0.918206857$, $b_5=-0.756704078$, $b_6=0.482199394$, $b_7=-0.193527818$, $b_8=0.035868343$.

The incomplete Gamma function is defined according to

$$\gamma(a,x)=\int_0^x t^{a-1}\exp(-t)\,\mathrm{d}t. \tag{A.4}$$

This was evaluated according to the method of Lindstrom [23] as,

$$\gamma(a,x)=\frac{x^a \mathrm{e}^{-x}}{a}\left(1+\sum_{i=1}^{n}\frac{x^i}{(a+i)!}\right). \tag{A.5}$$

Enumerated values of Γ and γ were found to agree precisely with those of Pearson and Hartley [22].

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