



NRC Publications Archive Archives des publications du CNRC

Fractal geometry and multimedia retrieval: a theoretical framework Paquet, Eric; Viktor, Herna L.

This publication could be one of several versions: author's original, accepted manuscript or the publisher's version. /
La version de cette publication peut être l'une des suivantes : la version prépublication de l'auteur, la version
acceptée du manuscrit ou la version de l'éditeur.

Publisher's version / Version de l'éditeur:

*3D Image Processing (3DIP) and Applications 2012 Conference Proceedings,
2012-02*

NRC Publications Record / Notice d'Archives des publications de CNRC:

<https://nrc-publications.canada.ca/eng/view/object/?id=5ad6b425-48bd-4c37-94fd-597aa8bdf79d>
<https://publications-cnrc.canada.ca/fra/voir/objet/?id=5ad6b425-48bd-4c37-94fd-597aa8bdf79d>

Access and use of this website and the material on it are subject to the Terms and Conditions set forth at

<https://nrc-publications.canada.ca/eng/copyright>

READ THESE TERMS AND CONDITIONS CAREFULLY BEFORE USING THIS WEBSITE.

L'accès à ce site Web et l'utilisation de son contenu sont assujettis aux conditions présentées dans le site

<https://publications-cnrc.canada.ca/fra/droits>

LISEZ CES CONDITIONS ATTENTIVEMENT AVANT D'UTILISER CE SITE WEB.

Questions? Contact the NRC Publications Archive team at

PublicationsArchive-ArchivesPublications@nrc-cnrc.gc.ca. If you wish to email the authors directly, please see the first page of the publication for their contact information.

Vous avez des questions? Nous pouvons vous aider. Pour communiquer directement avec un auteur, consultez la première page de la revue dans laquelle son article a été publié afin de trouver ses coordonnées. Si vous n'arrivez pas à les repérer, communiquez avec nous à PublicationsArchive-ArchivesPublications@nrc-cnrc.gc.ca.



Fractal Geometry and Multimedia Retrieval: a Theoretical Framework

Eric Paquet^{a, b}, Herna L. Viktor^{*b}

^aNational Research Council, 1200 Montreal Road, Ottawa, ON, K1A 0R6, Canada

^bSchool of Electrical Engineering and Computer Science, University of Ottawa, 800 King Edward, Ottawa, ON, K1N 6N5, Canada

ABSTRACT

This paper presents a theoretical analysis of the occurrence of fractal geometry within index spaces and discusses the impact for multimedia retrieval. Firstly, we explain how to detect the presence of such a fractal geometry. Then, with the fractal hypothesis in hand, we analyze the impact of this geometry when calculating the distance between indexes and searching for similar multimedia objects. We demonstrate that it is possible to define probability distributions in fractal index space and we present an approach to calculate them. Practical consequences are discussed, with particular emphasis to multimedia retrieval methods based on Bayesian analysis.

Keywords: Bayesian, distance, distribution, fractal, geometry, indexing, Lagrangian, multimedia, object, probability, retrieval

1. INTRODUCTION

When indexing a multimedia database, we usually associate an index, descriptor or feature vector to each object belonging to the database. The set of all indexes associated with a database forms a geometrical space [1]. It follows that it is important to understand the underlying geometry associated with these index spaces, because this geometry determines both the way distances are measured as well as the way probability distributions are defined. The distance measures are important for indexes comparison and retrieval. The distributions, on the other hand, are of prime importance when employing probabilistic retrieval methods, in particular for methods based on Bayesian analysis [2]. To date, relatively little attention has been directed toward studying the properties of this distribution, and associated impact thereof. To this end, [3] proposed a Riemannian space for 3D objects defined with level curves in which the distance is measured in terms of geodesics. Also, [4] constructed a Riemannian space by imposing ad hoc constraints on the metric based on predetermined geometrical transformations. Recently, [1] proposed a general analysis of index spaces in terms of Riemannian geometry. In this communication, we extend our reflection to index spaces that possess a fractal geometry. In particular, we demonstrate that a probability distribution for the indexes may be defined and we show how it may be calculated. Our approach is based on a remarkable theory of fractal spaces pioneered by Nottale [5, 6] which we shall follow very closely in our paper. Since this theory is practically unknown in the content-based indexing and retrieval field, we shall present an extensive description of it. Our paper is organized as follows. Section 2 provides an introduction to fractal spaces. In Section 3, we demonstrate how the probability distribution of indexes may be calculated when a fractal geometry is present. Section 4 shows the implications of fractal geometry for multimedia indexing and retrieval. Section 5 presents some experimental evidences that a fractal geometry can indeed be associated with an index space. This is followed by a conclusion.

2. FRACTAL GEOMETRY AND MULTIMEDIA INDEX SPACES

Most index spaces are discrete, in the sense that each multimedia object (e.g. image, 3D model) is associated with an index, which in turn corresponds to a point in the multidimensional index space. Consequently, for the purposes of this communication, the first step is to determine whether or not such a discrete set may be considered to be fractal. This may be achieved by calculating the fractal dimension of the discrete set [7]. Many definitions have been proposed in the literature, including the box counting dimension, the Rényi dimension, etc. [7]. For index spaces, one of the most

suitable approaches is the one based on correlation, since it has been found to be computationally suitable for such discrete sets [8]. The correlation is defined as

$$\gamma(\rho) \approx \frac{1}{n^2} \sum_{ij} H(\rho - r_{ij}) \quad (1)$$

where $H(x)$ is the Heaviside step function, r_{ij} is the Euclidian distance in between index i and j and, ρ is a particular distance scale. The fractal dimension is then obtained by calculating the logarithmic slope of the correlation function

$$D_F = \frac{\partial \ln(\gamma(\rho))}{\partial \ln \rho} \quad (2)$$

Research has shown that the behaviour of fractal spaces depends on the scale at which they are observed [7]. For instance, the length of a curve $Q(q, \varepsilon)$ is a function of the non-fractal or classical distance q (which can be any coordinate x^μ) and of the scale ε . The implication for multimedia retrieval is that the distance in between two indexes is a function of the scale. The equation governing the length of such a curve in fractal space is given by

$$\frac{\partial Q(q, \varepsilon)}{\partial \ln \varepsilon} = \beta(Q(q, \varepsilon)) \quad (3)$$

where $\beta(Q)$ is an arbitrary function of the fractal length. Equation (3) may be linearized, which means that the dependence on q may be ignored; if one considers a curve of infinitesimal length

$$\frac{dQ}{d \ln \varepsilon} \simeq \tau_F (Q_0 - Q) \quad (4)$$

where S_0 and τ_F are constants. After integration, one obtains

$$Q(\varepsilon) = Q_0 \left(1 + \left(\frac{\lambda}{\varepsilon} \right)^{\tau_F} \right) \quad (5)$$

where λ is an integration constant that represents a characteristic scale. Equation (5) may be generalized by making the fractal length explicitly dependant on the classical length [5, 6]:

$$Q(q, \varepsilon) = Q_0(q) \left(1 + \zeta(q) \left(\frac{\lambda}{\varepsilon} \right)^{\tau_F} \right) \quad (6)$$

where $\zeta(q)$ has a zero mean and an unitary variance but otherwise has an arbitrary form. Furthermore, Mandelbrot has demonstrated [5] that the exponent τ_F is related to the fractal and topological dimensions by

$$\tau_F = \frac{d \ln Q}{d \ln \left(\frac{\lambda}{\varepsilon} \right)} = D_F - D_T \quad (7)$$

Starting from Equation (6), we may obtain the infinitesimal distance in between two points in the index space from the stochastic mean of the following quadratic form

$$dS^2 = \sum_{\substack{\mu=1 \\ \nu=1}}^{D_T} \langle dX^\mu g_{\mu\nu} dX^\nu \rangle \equiv \sum_{\mu\nu} \langle dX^\mu g_{\mu\nu} dX^\nu \rangle \quad (8)$$

where D_T is the topological dimension of the index space, $g_{\mu\nu}$ is the constant diagonal metric associated with the classical coordinates e.g. the identity matrix in Euclidian space, where $\langle \bullet \rangle$ is the stochastic average and where dX^μ are the generalized coordinates defined as

$$dX^\mu = dx^\mu + d\xi^\mu \quad (9)$$

where x^μ are the classical or non-fractal coordinates. If the following correspondence is established

$$Q \sim X^\mu \quad q \sim x^\mu \quad \varepsilon \sim ds \quad (10)$$

one obtains for the generalized coordinates

$$\boxed{X^\mu = x^\mu \left(1 + \zeta \left(\frac{\lambda}{ds} \right)^{\tau_F} \right)} \quad (11)$$

From Equation (2) and the knowledge of the topological dimension, it is possible to calculate τ_F with Equation (7) and to obtain the length of curve in fractal space by integrating Equation (8); which is a generalization of the geodesic distance. From the above, one may deduce that such a geodesic distance is scale dependent if the index space is fractal from Equations (6), (8) and (11). In other words, when comparing indexes, the distance should always be measured at the same scale; distances measured at different scales are inconsistent.

3. PROBABILITY DISTRIBUTIONS ASSOCIATED WITH FRACTAL SPACES

In this section, we shall review the most relevant aspects of Nottale's theory [5, 6] for our purpose. Many indexing and retrieval methods are based on Bayesian analysis [2]. In order to apply such a probabilistic approach, it is important to be able to associate a probability distribution with the indexes. This distribution should not be *ad hoc*, but reflects the real distribution of the indexes. In this section, we explore the possibility of associating a probability distribution to an index space presenting a fractal geometry and we show how it may be calculated. Section 3.1 addressed the simplest case of an underlying geometry that is Euclidian. Sections 3.2 and 3.3 generalize the results of Section 3.1 to include spaces with, respectively, constant diagonal metrics and variable positive-definite metrics. Fractals are famous for self-similarity, in that the same basic structure repeats itself at various scale [7]. What is lesser known, though, is that the very notion of differentiability does not apply to fractal spaces [5, 6]. This is due to the fact that the left and the right derivatives are not identical, i.e.

$$\lim_{\delta x \rightarrow 0} \frac{f(x) - f(x - \delta x)}{\delta x} \neq \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad (12)$$

Consequently, following Equations (6) and (11), an infinitesimal element dX_\pm may be decomposed into a classical differential element $d_\pm x$, related to the underlying non-fractal geometry, and a fractal differential element $d\xi_\pm$:

$$\boxed{dX_\pm = d_\pm x + d\xi_\pm} \quad (13)$$

where the \pm refers to the right and left differential elements respectively, $x_\pm = [x_\pm^\mu]^\top = [x_\pm^1 \dots x_\pm^{D_T}]^\top$ with an analogous definition for ξ_\pm . Let us assume that the indexes may evolve in time. (Note that such a pseudo-time is introduced in order to obtain dynamical equations which are easier to manipulate. The proper final result shall be obtained by taking the static limit.) The evolution of the classical differential element is given by the product of its speed and the infinitesimal time

$$d_\pm x = v_\pm dt \quad (14)$$

while the evolution of the fractal differential element is given by

$$d\xi_{\pm} = \zeta_{\pm} \sqrt{2\mathcal{D}} \left(dt^2\right)^{\frac{1}{2D_F}} = \zeta_{\pm} \sqrt{2\mathcal{D}} dt^{\frac{1}{2}} \Big|_{D_F=2} \quad (15)$$

where Equation (15) comes from (11), where ζ_{\pm} are stochastic variables of zero mean and unitary variance

$$\boxed{\langle \zeta_{\pm} \rangle = 0 \quad \langle \zeta_{\pm}^2 \rangle = 1} \quad (16)$$

and where $\mathcal{D} = \frac{1}{2} \tau_F v^2$ is a parameter related to the particular fractal geometry. In order to derive a final analytical expression, we chose $D_F = 2$ in Equation (15). However, our approach is applicable to other fractal dimensions but it is not possible then to obtain a closed-form equation. Even if a derivative in the classical sense cannot be defined because of Equation (12). Further, the reader is reminded that a derivative can only be defined if the left derivative is equal to the right derivative, a fractal derivative may be defined by considering independently the left and right derivatives and by combining them with the imaginary number $i = \sqrt{-1}$ as a complex derivative

$$\boxed{\frac{D}{dt} \triangleq \frac{1}{2} \left(\frac{d_+}{dt} + \frac{d_-}{dt} \right) - \frac{i}{2} \left(\frac{d_+}{dt} - \frac{d_-}{dt} \right)} \quad (17)$$

where $\frac{D}{dt}$ is the fractal derivative. With the fractal derivative, one calculates the fractal speed, which may be decomposed into a real and imaginary part

$$\mathcal{V} \triangleq \frac{D}{dt} x = \frac{v_+ + v_-}{2} - i \frac{v_+ - v_-}{2} \quad (18)$$

where $x = [x^1 \dots x^{D_T}]^T$, $\mathcal{V} = [\mathcal{V}^1 \dots \mathcal{V}^{D_T}]^T$, etc. The total classical temporal derivative may be related to the classical-fractal coordinate with a Taylor expansion and a stochastic mean

$$\boxed{\frac{df}{dt} = \left\langle \frac{\partial f}{\partial t} + \sum_{\mu} \frac{\partial f}{\partial x^{\mu}} \frac{dX^{\mu}}{dt} + \frac{1}{2} \sum_{\mu\nu} \frac{\partial^2 f}{\partial x^{\mu} \partial x^{\nu}} \frac{dX^{\mu} dX^{\nu}}{dt} \right\rangle \Big|_{D_F=2}} \quad (19)$$

where f is an arbitrary function and $\langle \bullet \rangle$ is the stochastic average evaluated at $D_F = 2$. The number of terms in the expansion is finite if the fractal dimension is an integer [5, 6]; otherwise it is infinite and the expansion must be truncated to obtain a closed-form solution. From Equations (13), (15) and (16), the stochastic average of the coordinates are

$$\langle dX \rangle = dx \quad (20)$$

$$\langle d\xi^2 \rangle = 2\mathcal{D}dt \quad (21)$$

It follows that Equation (19) reduces to

$$\frac{d_{\pm} f}{dt} = \left(\frac{\partial}{\partial t} + v_{\pm} \nabla \pm \mathcal{D} \Delta \right) f \quad (22)$$

which allows us to express the fractal derivative (17) as

$$\boxed{\frac{D}{dt} \equiv \frac{\partial}{\partial t} + \hat{\mathcal{V}} \cdot \nabla} \quad (23)$$

where ∇ the D_T -dimensional gradient operator, $\hat{\mathcal{V}} \cdot \nabla$ is the inner Euclidian product and where the generalized fractal velocity is defined as

$$\boxed{\hat{\mathcal{V}} = \mathcal{V} - i\mathcal{D}\nabla} \quad (24)$$

Starting from the fractal velocity, it is possible to define a Lagrangian (a scalar function that characterizes the index space) [5, 6, 8] which, in its simplest expression, is limited to the kinetic energy

$$\mathcal{L}(x, V, t) = \frac{1}{2} \mu \mathcal{V}^2 \quad (25)$$

where μ is a unit dependent constant which may be chosen equal to unity with a proper choice of units. From the Lagrangian (25), one may calculate an action [5, 6, 8]

$$\mathcal{A} = \int_{t_0}^t \mathcal{L}(x, \mathcal{V}, t') dt' \quad (26)$$

The equations associated with the action (principle of least action) are obtained from the well-known [8] Euler-Lagrange equations which generalizes in the presence of fractal geometry as (notice the fractal derivative):

$$\boxed{\frac{D}{dt} \frac{\partial \mathcal{L}}{\partial \mathcal{V}} - \frac{\partial \mathcal{L}}{\partial x} = 0} \quad (27)$$

while the momentum associated with the Lagrangian is

$$\mathcal{P} \triangleq \frac{\partial \mathcal{L}}{\partial \mathcal{V}} = \mu \mathcal{V} = \nabla \mathcal{A} \quad (28)$$

From the action (26), we also define the scalar field

$$\psi = \exp\left(\frac{i\mathcal{A}}{\mathcal{A}_0}\right) \quad (29)$$

where \mathcal{A}_0 is a constant. Then, from Equations (28) and (29), one may express the fractal velocity in terms of this scalar field

$$\mathcal{V} = -i \frac{\mathcal{A}_0}{\mu} \nabla \ln \psi \quad (30)$$

If one evaluates the Euler-Lagrange equations (27) with the Lagrangian (25) one obtains

$$\mu \frac{D\mathcal{V}}{dt} = 0 \quad (31)$$

where \mathcal{V} , the fractal velocity was defined in (18) and where the fractal derivative was defined in (23). Finally, if one substitutes equation (30) into (31)

$$i\mathcal{A}_0 \frac{d}{dt} (\nabla \ln \psi) = 0 \quad (32)$$

and makes usage of the vectorial identity

$$\nabla \left(\frac{\Delta \psi}{\psi} \right) = 2 (\nabla \ln \psi \cdot \nabla) (\nabla \ln \psi) + \Delta (\nabla \ln \psi) \quad (33)$$

where Δ is the D_T -dimensional Laplace operator one obtains

$$\nabla \left(i \frac{\partial \ln \psi}{\partial t} + \mathcal{D} \frac{\Delta \psi}{\psi} \right) = 0 \quad (34)$$

which reduces to

$$\boxed{\mathcal{D}\Delta\psi + i\frac{\partial\psi}{\partial t} = 0} \quad (35)$$

which is the Schrödinger equation [8]. It has been established [8] that the scalar field ψ associated with this equation has an interpretation in terms of probability p ; the latter being related to the former by

$$\boxed{p(x) = \frac{\|\psi(x)\|^2}{\int \|\psi(x)\|^2 d^{D_F}x}} \quad (36)$$

and that $\psi(x)$ satisfies

$$\|\psi(x)\|^2 \geq 0 \quad (37)$$

i.e. that $\psi(x)$ is semi-positive definite. Consequently, we have associated a probability to the fractal geometry though Equations (36), (29), (26) and (25). Since the geometry of the indexes is static (at least, that is what we assume in the present communication), one should consider the stationary Schrödinger equation in lieu of (35)

$$\Delta\psi = 0 \quad (38)$$

i.e. that the scalar field ψ should be harmonic. This implies that any harmonic function is compatible with the fractal geometry and may be utilised to define the probability (36). From the previous section, it may be seen that the fractal geometry of the indexes determines, uniquely, the equation governing their probability distribution for a given fractal dimension. Other equations may be obtained for other fractal dimensions but an analytical expression (closed-form) is restricted to integer fractal dimensions because of Equation (19). In the previous section, an external pseudo time was introduced, in order to facilitate the calculations. In this section, we shall restrict ourselves to intrinsic (consequently, not arbitrary) quantities that are strictly defined within the fractal index space. To this end, the extrinsic time is replaced by an infinitesimal curve of intrinsic length ds (in classical coordinates). Such a quantity is related to the distance in between indexes, i.e. to their similarity. With such a choice, Equations (14) and (15) become

$$dX_{\pm}^{\mu} = d_{\pm}x^{\mu} + d\xi_{\pm}^{\mu} = v_{\pm}^{\mu}ds + \sqrt{2\lambda} \zeta_{\pm}^{\mu} (ds^2)^{1/2D_F} \quad (39)$$

where ζ_{\pm}^{μ} are stochastic variables of zero mean and unitary variance responsible for the fractal behaviour and where the non-fractal or classical velocity is defined as

$$v_{\pm}^{\mu} = \frac{d_{\pm}x^{\mu}(s)}{ds} = \lim_{\delta s \rightarrow 0^{\pm}} \left\langle \frac{x^{\mu}(s + \delta s) - x^{\mu}(s)}{\delta s} \right\rangle \quad (40)$$

Here we choose gain $D_F = 2$. However, the same procedure may also be applied to other fractal dimensions; the final expression might not have a closed-form, though. One should notice that the vectorial notation has been replaced by a tensorial notation in order to make the metric explicit. The definition of the fractal derivative is similar to the one of Equation (17) where the extrinsic pseudo time is replaced by the intrinsic infinitesimal length

$$\boxed{\frac{D}{ds} \triangleq \frac{(d_{+} + d_{-}) - i(d_{+} - d_{-})}{2 ds}} \quad (41)$$

where $\frac{D}{ds}$ is the intrinsic fractal derivative. From this derivative, one may define a fractal velocity

$$\mathcal{V}^{\mu} \triangleq \frac{Dx^{\mu}}{ds} = \frac{v_{+}^{\mu} + v_{-}^{\mu}}{2} - i \frac{v_{+}^{\mu} - v_{-}^{\mu}}{2} \quad (42)$$

where v_{\pm}^{μ} are defined in Equation (40). In the previous section, we implicitly assumed that the classical space is Euclidian which means that the inner product, the gradient operator and the Laplace operator are Euclidian and the metric is the identity matrix. In this section, we relax this assumption by allowing arbitrary constant positive-definite diagonal metrics. Since $\eta^{\mu\nu}$ is a metric, the following relations also apply

$$\sum_{\kappa} \eta^{\mu\kappa} \eta_{\kappa\nu} = \delta_{\nu}^{\mu} \quad (43)$$

$$\mathcal{Q}^{\mu} = \sum_{\nu} \eta^{\mu\nu} \mathcal{Q}_{\nu} \quad (44)$$

for the inverse of the metric and the transformation in between covariant and contravariant tensorial indices, respectively. The stochastic relations (16) may be rewritten with the help of the metric as

$$\left\langle d\xi_{\pm}^{\mu} d\xi_{\pm}^{\nu} \right\rangle = \pm \lambda \eta^{\mu\nu} ds \quad (45)$$

where λ is a constant; remember that the metric is diagonal. The calculation of the probability distribution then follows a procedure analogous to the one in the previous section. From expansion (19) with dt replaced by ds and Equation (45) one obtains

$$\frac{d_{\pm} f}{ds} = \frac{\partial}{\partial s} + \sum_{\mu} \left(v_{\pm}^{\mu} \partial_{\mu} \pm \frac{1}{2} \lambda \partial^{\mu} \partial_{\mu} \right) f \quad (46)$$

which allows to write the fractal derivative (41) as

$$\frac{D}{ds} = \sum_{\mu} \left(\mathcal{V}^{\mu} - \frac{1}{2} i \lambda \partial^{\mu} \right) \partial_{\mu} \quad (47)$$

Following the same approach than in Section 3.1.2, it is possible to define a Lagrangian and an action that uniquely characterize the geometry of the index space [5, 6]. The dynamical equations associated with such an action are obtained from the Euler-Lagrange equations (i.e. the principle of least action). The action represents the fractal distance in between two arbitrary indexes. This intrinsic quantity is related to the concept of similarity, which is the most important geometrical aspect associated with the index space from a retrieval perspective. It is defined as

$$\mathcal{A} \triangleq -\chi \int \sqrt{\sum_{\mu} D x^{\mu} D x_{\mu}} \Rightarrow \mathcal{A} = -\chi \int \sum_{\mu} \mathcal{V}_{\mu} dx^{\mu} \quad (48)$$

where χ is a unit dependent parameter that may be set equal to unity by a proper choice of units. The momentum associated with the Lagrangian becomes

$$\mathcal{P}_{\mu} \triangleq \frac{\partial \mathcal{L}}{\partial \mathcal{V}^{\mu}} = \chi \mathcal{V}_{\mu} = -\partial_{\mu} \mathcal{A} \quad (49)$$

As in the previous section, one may define a scalar field

$$\psi = \exp \left(\frac{i \mathcal{A}}{\chi \lambda} \right) \quad (50)$$

Using once more the generalized (fractal derivative) Euler-Lagrange equations

$$\frac{D}{ds} \frac{\partial \mathcal{L}}{\partial \mathcal{V}^{\mu}} - \frac{\partial \mathcal{L}}{\partial x^{\mu}} = 0 \quad (51)$$

with the Lagrangian defined in (48) one obtains

$$\frac{\mathcal{D} \mathcal{V}_{\mu}}{ds} = -i \lambda \sum_{\nu} \left(\mathcal{V}^{\nu} \partial_{\nu} - \frac{1}{2} i \lambda \partial^{\nu} \partial_{\nu} \right) \partial_{\mu} (\ln \psi) = 0 \quad (52)$$

which may be simplified to

$$\partial_\mu \left(\frac{\lambda^2}{\psi} \sum_\nu \partial^\nu \partial_\nu \psi \right) = 0 \quad (53)$$

which reduces to the Klein-Gordon [8] equation:

$$\boxed{\lambda^2 \sum_\mu \partial^\mu \partial_\mu \psi = \psi} \quad (54)$$

This equation may also be interpreted in terms of probabilities [8] if the metric is positive-definite (which is the case here by hypothesis) i.e. if

$$\sum_{\mu\nu} x^\mu \eta_{\mu\nu} x^\nu > 0 \quad \forall x \quad (55)$$

Then, the probability is obtained from Equation (36). If Equation (55) is not satisfied, negative probabilities will appear which are incompatible with the very definition of a probability. It should be noted that Equation (54) is the Eigen equation of the Laplace operator: ψ being the Eigen function and $1/\lambda^2$ being the Eigen values which shows once more the importance of the Laplacian in the definition of a probability in a fractal space. We see that with the sole assumption of a fractal dimension of two (2), and by using only intrinsic quantities, the index space is characterized by the spectral decomposition of the Laplacian. Such a decomposition extensively appears when working with the heat equation and diffusion maps [9, 10] for 3D object retrieval but emerges here directly from the underlying fractal geometry.

4. IMPLICATIONS OF FRACTAL GEOMETRY OF 3D OBJECT INDEXING AND RETRIEVAL

What, then, are the practical implications of a fractal structured index space? The presence of such a structure has far-reaching consequences for multimedia object indexing and retrieval, especially in terms of similarity and probability. (Note that the presence of fractal structure may be determined using Equations (1) and (2).) The first consequence is that the concept of distance in between two indexes (and consequently, the concept of similarity in between two multimedia objects) depends on the scale at which the distance is measured, as was shown in Section 2. Such dependence of the distance on the scale is present irrespectively of the way the distance in between indexes is defined. In practice, this means that if a fractal structure is present, distances should be defined at a particular scale or resolution. The corollary is that only distances that are measured at the same scale are comparable. It does not make any sense to compare distances or similarity measures at different scales. That is, multiple scales may be utilised as long as they are consistent. In Section 3, we have shown that it is possible to define a probability, actually a family of probability distributions, on the index space. This is by far not a trivial result. Such a probability distribution may be defined from harmonic functions, i.e. the solutions of the Laplacian as demonstrated in Section 3. An alternative is to use the spectrum of the Laplacian, meaning the Eigen functions associated with the Laplacian (generalization of the previous case). Thus, the Laplacian seems to play a central role in the definition of probability in fractal index space as it does, for instance, for 3D indexing based on the heat equation [10]. If the index space is a bounded domain, it may be shown [8] that the Eigen functions of the Laplacien form an orthonormal basis (in a Hilbert space). This provides us with a method to define probability distributions which are compatible with the underlying fractal geometry of the indexes. In this case, there still remains huge degrees of freedom for their definition, in the sense that there are no *a priori* theoretical restrictions on which solutions of the Laplace Eigen equation should be chosen. Since the Laplacian is a linear operator, it is possible to express ψ as a combination of linear harmonic functions since both the Schrödinger and the Klein-Gordon equations are linear. The probability distributions that we have defined in this paper are of prime importance for retrieval methods based on Bayesian analysis [2]. Indeed, the prior, as any other probability, must be defined while taking the background, or known, information into account [2]. If a particular piece of information is available, it must be incorporated into the definition of the probability distribution. If it is not incorporated, the model may become *ad hoc* and the conclusions, although obtained by strict application of the Bayesian rules, may become unreliable. In other words, Bayesian analysis cannot do better than the prior. Consequently, if the index space has a fractal structure, it is necessary to choose a probability distribution which is compatible with this structure; that is, chosen according to the approach presented in Section 3. At this point, an important question remains, namely are there other ways to define the distributions which

were not covered in Section 3 but nevertheless would be compatible with the fractal structure of an index space? The Lagrangian defined in Equations (25) and (48) are the simplest that could be defined. For this reason, they are the most likely ones that are the most commonly used in practical applications. Nevertheless, nothing prevents us to define a more complex Lagrangian if additional background information points in that direction. For instance, one may associate a potential $\Phi(x)$, that is to say a scalar function, that associates a certain energy or weight to each index. This function is related to the intrinsic (emission) probability associated with x i.e. to the probability of obtaining x irrespectively of the underlying geometry. In other words, we assume that x is a random variable. Indeed, it can be shown [9], that the potential is related to the intrinsic probability by

$$\Phi(x) = -\log(P(x)) \quad (56)$$

In this case, the Lagrangian, Equation (25) transforms as

$$\mathcal{L}(x, V, t) = \frac{1}{2}\mu\mathcal{V}^2 + \Phi(x) \quad (57)$$

and the equation of the scalar field associated with the probability distribution ψ becomes

$$\boxed{\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{1}{2\mu}\Phi(x)\psi = 0} \quad (58)$$

which is also Schrödinger equation and consequently is compatible with the definition of a probability [8]. A similar generalization holds for the Klein-Gordon equation. In order to obtain probability, one must either solve the Schrödinger equation with potential or the Klein-Gordon equation with potential and then compute the probability with the help of Equation (36). At this stage, one might ask if there is any experimental evidences of a fractal geometry at the level of an index space. Although the objective of this paper is to present a theoretical framework, we would like to support our development with some experimental evidences. These evidences are by no mean a proof but rather an indication that a fractal geometry is possible at the index space level. The details related to this experiment may be found in [11]. The experiment may be outlined as follows. A high intensity pulsed laser beam is focused on a low pressure xenon gas, the integrated intensity is measured as the laser beam exits the interaction chamber. This experiment simulates, for instance, the scanning of a fractal surface with a laser beam (shape acquisition). The integrated intensity of the pulse on the sensor is mathematically equivalent to the geometrical moment of order zero of the pulse which is one of the simplest descriptor one may conceive.

$$M_{00} = \iint dx dy x^0 y^0 I(x, y) \quad (59)$$

From the sequence of intensity, a Poincaré map [11] is constructed and the fractal dimension is computer with the help of Equations (1) and (2). The main conclusions of the experiment are the following. The moment of order zero space does have a fractal dimension: in this particular case 1.73. As shown by a simulation, this fractal dimension is not related to a possible random fluctuation of the intensity nor to the precision or the accuracy involved in the measurement process. This experiment demonstrates that index spaces with fractal geometry do indeed exist!

5. CONCLUSIONS

This paper explored the theoretical and practical implications of a possible fractal structure of spaces associated with multimedia object indexes. We have shown that the distance, or the similarity, in between two indexes may only be defined at a particular scale, or level of detail. In addition, various indexes should only be compared at the same scale or at the same set of scales. We have also shown that it is possible to define a probability distribution even if the index space has a underlying fractal geometry. We proposed various procedures for this calculation. Future work will involve a study of the impact of a combined fractal and Riemannian structure of the index space. We aim to apply the theory to real cases and studying the implications for cluster analysis. We will also pursue analytical solutions for other fractal dimensions.

REFERENCES

- [1] Paquet, E. and Viktor, H. L., "Index Spaces for 3D Retrieval: Toward a Better Understand of their Geometry and Probability Distribution," 3D Image Processing and Applications (3DIP), IS&T/SPIE International Symposium on Electronic Imaging, Proc. SPIE Vol. 7526, January 18-19, San Jose, CA, USA, 75260J-1 - 75260J-10 (2010).
- [2] Chen, Y. and Cipolla, R., "Learning Shape Prior for Single View Reconstruction," 3-D Digital Imaging and Modeling (3DIM), October 3-4, Kyoto, Japan, 1425-1432 (2009).
- [3] Samir C., Srivastava A., Daoudi M. and Klassen E., "An Intrinsic Framework for Analysis of Facial Surfaces," International Journal on Computer Vision 82(1), 80-95 (2009).
- [4] Kilian, M., Mitra, N. J. and Pottmann, H., "Geometric Modeling in Shape Space," ACM Transactions on Graphics 26(3), 8 pp. (2007).
- [5] Nottale, L., Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity, World Scientific, Singapore (1993).
- [6] Nottale, L., Theory of Scale Relativity, the: Fractal Space-Time, Nondifferentiable Geometry and Quantum Mechanics, World Scientific, Singapore (2011).
- [7] Hilborn, R.C., Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers, Oxford University Press, Oxford (2000).
- [8] Frankel, T, The Geometry of Physics: An Introduction, Cambridge University Press, Cambridge (2003).
- [9] Nadler B., Lafon, S., Coifman, R. R. and Kevrekidis, I. G., "Diffusion maps, spectral clustering and reaction coordinates of dynamical systems," Applied and Computational Harmonic Analysis 21 (1), 113-127 (2006).
- [10] Ovsjanikov, M., Bronstein, A.M., Bronstein, M.M. and Guibas, L.J., "Shape Google: a computer vision approach to isometry invariant shape retrieval," IEEE 12th International Conference on Computer Vision Workshops, September 27-October 4, Kyoto, Japan, 320-327 (2009).
- [11] Paquet, E. and Chin, S. L., " Fractal geometry and multiphoton ionization," J. Phys. B: At. Mol. Opt. Phys. 24, L579-L582 (1991).