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DIRECT DERIVATION OF RADIATION RESISTANCE OF A VIBRATING PANEL

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LA DETERMINATION DIRECTE DE LA RESISTANCE D'UN PANNEAU VIBRANT AU RAYONNEMENT

SOMMAIRE

L'auteur se sert de la formule de rayonnement de Rayleigh afin de déterminer la résistance d'un panneau vibrant au rayonnement. On compare les résultats à ceux qu'ont obtenus d'autres auteurs en utilisant des analyses différentes, à savoir une méthode de transformation et une méthode d'oscillateur accouplé.

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DIRECT DERIVATION OF RADIATION RESISTANCE OF A VIBRATING PANEL

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Rayleigh's radiation formula is used to derive the radiation resistance of a vibrating panel. The result is compared to those obtained by other authors using different analyses: viz., a transform method and a coupled oscillator method.

1. INTRODUCTION

This paper presents a direct derivation of the radiation resistance of a vibrating plate, by use of Rayleigh's radiation formula. Two earlier studies, one by Maidanik [1] and the other by Lyon and Maidanik [2] have obtained the radiation resistance, in one case by a transform method and in the other by a coupled oscillator approach. Both these techniques agree in the final general form of the expression for the radiation resistance, but differ by a factor of eight. It is found that the direct method to be described agrees with the transform technique. Further examination of the coupled oscillator method indicates two apparent errors and when these are corrected all three methods agree.

In the present paper the direct derivation is first described, and subsequently the relations between the three methods are discussed.

2. DIRECT DERIVATION

Consider the far field velocity potential ΔV produced by two elements at $\mathbf{x}_1$ and $\mathbf{x}_2$ vibrating with velocities $u(\mathbf{x}_1)$ and $u(\mathbf{x}_2)$, respectively, in an infinite baffle:

$$\Delta V = u(\mathbf{x}_1)\,\Delta(\mathbf{x}_1)\,(2\pi r_1)^{-1}\exp(-jkr_1) + u(\mathbf{x}_2)\,\Delta(\mathbf{x}_2)\,(2\pi r_2)^{-1}\exp(-jkr_2), \tag{1}$$

where r_1 and r_2 are the distances of the two elements from the observation point (R, θ, φ_0) and k is the wave number in the medium. The coordinates R, θ and φ_0 are the usual spherical coordinates where θ is measured from the normal to the vibrating surface (Figure 1).

The corresponding equation for the pressure Δp is given by

$$\Delta p = j\omega\rho\,\Delta V. \tag{2}$$

Throughout it will be assumed that the vibration of the plate is either modal or reverberant: i.e., the instantaneous or statistically expected product of the velocity terms is real. Thus

$$|\Delta p|^2 = \left(\frac{\omega\rho}{2\pi}\right)^2 \sum_S \left\{u^2(\mathbf{x}_1)\frac{(\Delta\mathbf{x}_1)^2}{r_1^2} + u^2(\mathbf{x}_2)\frac{(\Delta\mathbf{x}_2)^2}{r_2^2} + \right.$$
$$\left. + 2\cos\left[k(r_1 - r_2)\right] u(\mathbf{x}_1)\,u(\mathbf{x}_2)\,\Delta(\mathbf{x}_1)\frac{\Delta(\mathbf{x}_2)}{r_1 r_2}\right\}, \tag{3}$$

where the summation is carried out over the vibrating surface. Equation (3) simplifies to

$$|\Delta p|^2 = \left(\frac{\omega\rho}{2\pi}\right)^2 (R)^{-2} \iint_S u(\mathbf{x}_1)\, u(\mathbf{x}_2) \cos k(r_1 - r_2)\, \mathrm{d}\mathbf{x}_1\, \mathrm{d}\mathbf{x}_2, \tag{4}$$

where r_1 and r_2 have been replaced by R in the divergence term and the summation has been replaced by an integration. Since only the far field case is being considered $(r_1 - r_2)$ may be replaced by

$$[w_1 \cos(\varphi_1 - \varphi_0) \sin\theta - w_2 \cos(\varphi_2 - \varphi_0) \sin\theta],$$

where $w = |\mathbf{x}|$. Then the power W radiated over a half space will be given by

$$\begin{aligned} W &= \int \frac{|\Delta p|^2}{2\rho c} R^2\, \mathrm{d}\Omega \\ &= \left(\frac{\rho c k^2}{8\pi^2}\right) \iint_S \int_{\theta=0}^{\theta=\pi/2} \int_{\varphi=0}^{\varphi=2\pi} u(\mathbf{x}_1)\, u(\mathbf{x}_2) \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi_0 \cos\{k[w_1 \cos(\varphi_1 - \varphi_0) - w_2 \cos(\varphi_2 - \\ &\quad - \varphi_0)] \sin\theta\}\, \mathrm{d}\mathbf{x}_1\, \mathrm{d}\mathbf{x}_2, \end{aligned} \tag{5}$$

where Ω is a solid angle.

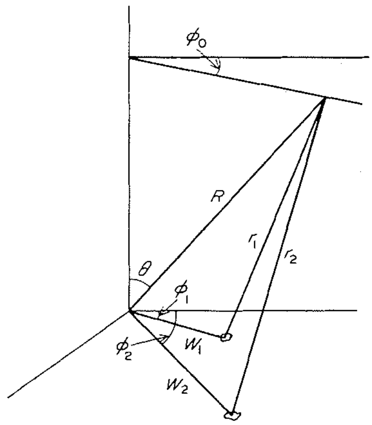

Figure 1. Coordinate system.

The cosine factor in the integrand of equation (5) can be written as

$$\begin{aligned} \cos[\ ] &= \mathrm{Re}[\exp\{\ \}] \\ &= \mathrm{Re} \sum_{m,\, n=0}^{\infty} \varepsilon_m \varepsilon_n i^{m-n} \cos m(\varphi_1 - \varphi_0) \cos n(\varphi_2 - \varphi_0) J_m(kw_1 \sin\theta) J_n(kw_2 \sin\theta), \end{aligned} \tag{6}$$

where $\varepsilon_m = 2$ for $m \neq 0$ and $\varepsilon_m = 1$ for $m = 0$. Substituting equation (6) into equation (5) and calculating the integral, I, of equation (5) with respect to φ_0 yields, for $m = n$,

$$\begin{aligned} I &= \pi \cos m(\varphi_1 - \varphi_2), & m \neq 0 \\ &= 2\pi, & m = 0. \end{aligned}$$

When $m \neq n$, $I = 0$. In other words

$$\begin{aligned} I\varepsilon_m \varepsilon_n &= 4\pi \cos m(\varphi_1 - \varphi_2) & \text{for } m \neq 0, \\ &= 2\pi & \text{for } m = 0, \end{aligned}$$

or

$$I\varepsilon_m \varepsilon_n = \varepsilon_m 2\pi \cos m(\varphi_1 - \varphi_2). \tag{7}$$

From Neumann's addition theorem,

$$J_0(kw' \sin\theta) = \sum_m \varepsilon_m J_m(kw_1 \sin\theta) J_m(kw_2 \sin\theta) \cos m(\varphi_1 - \varphi_2),$$

and so

$$W = \frac{\rho c k^2}{8\pi} \iint_S u(\mathbf{x}_1) u(\mathbf{x}_2) \int_0^{\pi/2} 2\pi J_0(kw' \sin\theta) \sin\theta \, d\theta \, d\mathbf{x}_1 \, d\mathbf{x}_2, \tag{8}$$

where

$$w' = [w_1^2 + w_2^2 - 2w_1 w_2 \cos(\varphi_1 - \varphi_2)]^{1/2}.$$

Substituting $k \sin\theta = z$ gives

$$W = \left(\frac{\rho c k^2}{8\pi^2}\right) \iint_S u(\mathbf{x}_1) u(\mathbf{x}_2) \int_0^k 2\pi J_0(w' z) \frac{z \, dz}{k\sqrt{k^2 - z^2}} d\mathbf{x}_1 \, d\mathbf{x}_2$$

$$= \frac{\rho c k^2}{4\pi} \iint_S u(\mathbf{x}_1) u(\mathbf{x}_2) \frac{\sin kw'}{kw'} d\mathbf{x}_1 \, d\mathbf{x}_2. \tag{9}$$

The radiation resistance R_{rad} is defined by $W = R_{rad}\langle v^2 \rangle$, where $\langle v^2 \rangle$ is the mean squared velocity of the surface with respect to space and time. If $u(x)$ is expressed in modal form (e.g., $u_0 \sin k_{0x} x \sin k_{0y} y$), then $\langle v^2 \rangle = u_0^2/8$ (provided that $k_{0x}, k_{0y} \neq 0$) and

$$R_{rad} = \frac{2\rho c k^2}{\pi} \iint_S \Psi(\mathbf{x}_1, \mathbf{x}_2) \Phi(\mathbf{x}_1, \mathbf{x}_2) \, d\mathbf{x}_1 \, d\mathbf{x}_2, \tag{10}$$

where $\Phi(\mathbf{x}_1, \mathbf{x}_2)$ is the modal vibration pattern and $\Psi(\mathbf{x}_1, \mathbf{x}_2)$ is the cross correlation coefficient of a three-dimensional random pressure field. If the panel is radiating energy from both sides, then

$$R_{rad} = \frac{4\rho c k^2}{\pi} \iint_S \Psi(\mathbf{x}_1, \mathbf{x}_2) \Phi(\mathbf{x}_1, \mathbf{x}_2) \, d\mathbf{x}_1 \, d\mathbf{x}_2. \tag{11}$$

By returning to equation (9), the two cases of small and very large vibrator dimensions compared to the wavelength can be considered. Thus, for a small vibrator,

$$W = \frac{\rho c k^2}{4\pi} \int u^2 \, dS^2 = \frac{\rho c k^2}{4\pi} u^2 S^2$$

and

$$R_{rad} = \frac{\rho c k^2}{2\pi} S^2. \tag{12}$$

Similarly, when the surface is very large, and the points on the surface are vibrating in unison, then it acts as a piston and equation (9) reduces to

$$W = \frac{\rho c k^2}{4\pi} \iint_S u^2 \frac{\sin kz}{kz} 2\pi z \, dz \, 2\pi r \, dr, \tag{13}$$

where, for simplicity, circular symmetry is assumed. In this case

$$R_{rad} = \rho c S.$$

It is of interest to calculate the radiation resistance for a reverberant field in the plate. Such a field would be produced by a point noise source acting in or on the plate. The plate is assumed to be large and the expected value of the product $u(\mathbf{x}_1)u(\mathbf{x}_2)$ is substituted into equation (9). In this case equation (9) can be written as

$$R_{\mathrm{rad}} = \frac{\rho c k_a^2}{2\pi} \iint_S J_0(k_b r) \frac{\sin k_a r}{k_a r} 2\pi r \, \mathrm{d}r \, 2\pi R \, \mathrm{d}R$$

$$= \frac{\rho c S}{\cos\varphi}, \tag{9a}$$

where $\cos\varphi = (k_a^2 - k_b^2)^{1/2}/k_a$ and where the velocity product is replaced by the two-dimensional correlation coefficient. In equation (9a), k_b is the wave number in the plate and k_a that in the surrounding medium.

3. TRANSFORM METHOD

Equation (9) can now be compared with the form derived by Maidanik [1] using a transform method. Maidanik gives as the radiated power, W,

$$W = \tfrac{1}{2}\rho c k_a \operatorname{Re}\left[\int_{-\infty}^{\infty} \mathrm{d}\mathbf{k}\, \nu(\mathbf{k})^2/(k_a^2 - \mathbf{k}^2)^{1/2}\right], \tag{14}$$

where

$$\nu(\mathbf{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\mathbf{x}) \exp(-i\mathbf{k}\mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{15}$$

and

$$|\nu(\mathbf{k})|^2 = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} u(\mathbf{x}_1)\, u(\mathbf{x}_2) \exp[i\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)] \, \mathrm{d}\mathbf{x}_1 \, \mathrm{d}\mathbf{x}_2. \tag{16}$$

Substituting equation (16) into equation (14) and considering only the integration I with respect to k gives

$$I = \operatorname{Re}\int_{-\infty}^{\infty} \exp[i\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)] \, \mathrm{d}\mathbf{k}/(k_a^2 - \mathbf{k}^2)^{1/2}. \tag{17}$$

Equation (17) is a version of the Sommerfeld integral and may be written as

$$I = 2\pi k_a \frac{\sin k_a w'}{k_a w'}. \tag{18}$$

Substituting from equations (18) and (16) into equation (14) yields equation (9) again.

4. COUPLED OSCILLATOR METHOD

Although the radiation method and the transform method give the same result for the radiation resistance they seem to differ from the coupled oscillator approach used by Lyon and Maidanik [2]. The radiation resistance by this method is said to be

$$R_{\mathrm{rad}} = \frac{16\rho c k_a^2}{\pi}\iint \mathrm{d}\mathbf{x}_1 \, \mathrm{d}\mathbf{x}_2 \, \Psi(\mathbf{x}_1, \mathbf{x}_2)\, \Phi(\mathbf{x}_1, \mathbf{x}_2), \tag{19}$$

which is eight times the value given by equation (10).

There seem to be two errors in the paper of Lyon and Maidanik, however, which account for this factor of eight. The following gives a brief outline of the relevant part of their paper and locates the suspected errors. The notation and equation number (prefixed by A) are those of the authors. The modal equations for the vibration surface(s) and the coupled ambience (q) are given as

$$\ddot{s}_m + \beta_m \dot{s}_m + \omega_m^2 s_m + B_{rm}\dot{q}_r = F_m - \sum_{k\neq r} B_{km}\dot{q}_k = F'_m, \tag{A9.8a}$$

$$\ddot{q}_r + \beta_r \dot{q}_r + \omega_r^2 q_r - B_{mr}\dot{s}_m = G_r - \sum_{n\neq m} B_{nr}\dot{s}_n = G'_r. \tag{A9.8b}$$

The power flow from the mth mode of the plate vibration to the rth mode of vibration of the surrounding medium is given by

$$j_{mr} = g_{mr}(\theta'_m - \theta'_r),$$

where θ'_m, θ'_r are equivalent "temperatures" given by

$$\theta'_m = \langle F'_m \dot{s}_m\rangle/\beta_m, \tag{A9.12}$$

$$\theta'_r = \langle G'_r \dot{q}_r\rangle/\beta_r, \tag{A9.13}$$

and g_{mr} is given by

$$g_{mr} = B_{mr}^2(\beta_m\omega_r^2 + \beta_r\omega_m^2) \div [(\omega_m^2 - \omega_r^2)^2 + (\beta_m + \beta_r)(\beta_m\omega_r^2 + \beta_r\omega_m^2)], \tag{A9.11}$$

where

$$B_{mr} = (\rho c^2/V\varepsilon_r M_m)^{1/2}\int_S \Psi(\mathbf{x})\,\Phi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{A9.6}$$

M_m is the modal mass; Φ_r and Ψ_m are the eigenfunctions for the plate and the surrounding medium such that

$$\int_r \Psi_r(\mathbf{x}_q)\,\Psi_k(\mathbf{x}_q)\,\mathrm{d}\mathbf{x}_q = V\varepsilon_r\,\delta(r,k), \tag{A7.7}$$

where $\varepsilon_r = 1/8$ if all indices of (r) are non zero, $\varepsilon_r = 1/4$ if two indices of (r) are non zero, and r stands for three eigennumbers.

As only the field modes lying in a narrow band about ω_m need be considered, equation (A9.11) may be integrated to obtain

$$\sum_r g_{mr} = B_{mr}^2 \frac{\pi}{2} n(\omega), \tag{20}$$

where $n(\omega)$ is the spectral density of field modes. Thus, equation (20) differs from that obtained by integrating equation (A9.17) of Lyon and Maidanik which is

$$\sum_r g_{mr} = B_{mr}^2 \pi n(\omega). \tag{A9.17}$$

Now, as

$$R_{\text{rad}} = M\sum_r g_{mr} \tag{A9.21}$$

$$= \frac{1}{2}\frac{\pi\rho c^2}{V\varepsilon_n\,\varepsilon_m} n(\omega) \int_S \Psi_r(\mathbf{x}_1)\,\varphi_m(\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_1 \int_S \Psi_r(\mathbf{x}_2)\,\varphi_m(\mathbf{x}_2)\,\mathrm{d}\mathbf{x}_2, \tag{21}$$

where

$$\varepsilon_m = M_m/M \tag{A8.7}$$

is the ratio of modal to total mass ($\varepsilon_m = 1/4$ for two dimensional vibration), thus

$$R_{\mathrm{rad}} = (4\pi\varepsilon_r\,\varepsilon_m)^{-1}\,\rho ck^2 \int_S \mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2\,\Psi(\mathbf{x}_1,\mathbf{x}_2)\,\Phi(\mathbf{x}_1,\mathbf{x}_2). \tag{22}$$

$\Psi(\mathbf{x}_1,\mathbf{x}_2)$ is defined as the product of cosine terms and its average value over the surface for $\mathbf{x}_1 = \mathbf{x}_2$ is 1/4. Thus $\Psi(\mathbf{x}_1,\mathbf{x}_2)$ should be written as $\sin kR/4kR$ and not $\sin kR/kR$, where $R = |\mathbf{x}_1 - \mathbf{x}_2|$, in equation (A7.16). Whence

$$R_{\mathrm{rad}} = \frac{2\rho ck^2}{\pi} \int_S \mathrm{d}\mathbf{x}_1\,\mathrm{d}\mathbf{x}_2\,\Psi(\mathbf{x}_1,\mathbf{x}_2)\,\Phi(\mathbf{x}_1,\mathbf{x}_2), \tag{23}$$

which is now the same as equation (10).

5. SUMMARY

Use of Rayleigh's radiation formula gives results in agreement with the known radiation resistance for very small sources acting in infinite baffles and for very large pistons where all the points on the vibrating surface are in phase. The results agree with those derived by Maidanik's transform method, but not those as quoted by Lyon and Maidanik using a coupled oscillator technique. By making two corrections to the coupled oscillation technique their results are brought into agreement with those obtained by the radiation formula and with those obtained by using a transform method of analysis.

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