Ground temperature studies at Saskatoon and Ottawa, Canada

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GROUND TEMPERATURE STUDIES AT SASKATOON AND OTTAWA, CANADA

BY

D. C. PEARCE

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by D.C. Pearce

ERRATA

Page		
280	eq. (1),	$a^n(x)$ should read $a_n(x)$.
283	eq. (5),	x_{li} should read x_{1i}.
"	line 23,	x_{li} should read x_{1i}.
"	line 24,	A_{12} should read A_{10}.
284	eq. (8)	π should read η.
285	Table I, 3rd column, u_o should read ${}_su_o$.	
286	Subheading, Appendix A, l.c. should read "a".	
287	eq. 2(a) should read, $f(u) = Ae^{\frac{\beta u}{\alpha}} + Be^{-\frac{\beta u}{\alpha}}$	
"	eq. (3a) should read $f(u) = Ce^{\beta u}$.	
289	eq. (3b), $\frac{\partial^2 v}{\partial x^2}(x,v)$	should read $\frac{\partial^2 v}{\partial x^2}(x,t)$.
"	eq. (5b)	$\frac{\partial x}{\partial v}$ should read $\frac{\partial v}{\partial x}$.
"	eq. (9b)	$\tan^{-1}(\frac{\theta}{H\theta})$ should read $\tan^{-1}(\frac{\theta}{1+\theta})$.
290	eq. (12b)	A_o should read a_o.
"	eq. (13b)	$\zeta(t)\ b_o$ should read $\xi(t).b_o$.
"	line 15	$u(t)$ should read $u_s(t)$.
"	8 lines from bottom, delete "a" in italics.	

Extrait des Comptes Rendus et Rapports — Assemblée Générale de Toronto 1957
(Gentbrugge 1958)
Tome IV, p. 279 à 290

GROUND TEMPERATURE STUDIES AT SASKATOON AND OTTAWA, CANADA

D. C. PEARCE

ABSTRACT

An analysis is presented of the annual ground temperature data obtained at Ottawa and Saskatoon, Canada, by the Division of Building Research of the National Research Council from 1952 to 1955. The annual ground temperature variation was observed to be consistent with the theory of heat flow in a homogeneous semi-infinite solid. The observations yielded an average soil thermal diffusivity which did not vary from year to year. It was possible to include a term which approximately accounted for the insulation effect of a snow cover. Additional calculation showed the order of magnitude of the deviations to be expected from variations in the soil thermal constants.

Shortly after the formation of the Division of Building Research of the National Research Council in 1947, a study of ground temperatures was undertaken because a knowledge of the ground temperature regime was necessary to cope with several of the problems brought to the attention of the Division. The first step in this program was to establish a number of field observation stations throughout Canada. A general description of these field studies has been published ([1]). The present paper describes a detailed analysis of the observations obtained from two of the stations: one on the Montreal Road property at Ottawa, Ontario, and the other at the Prairie Regional Station at Saskatoon, Saskatchewan. These records were selected from the total available because they are reasonably complete and they extend over a period of several years.

The installation at the Montreal Road Laboratories was established specifically to study ground temperatures. Thermocouples were installed in test pits of both sand and clay to a depth of 8 feet. Certain of the test pits were cleared of snow throughout the winter to evaluate the effect of the snow cover. The details of the Ottawa ground temperature installation have been published ([2]).

The installation at the Prairie Regional Station was part of an outdoor test hut study. In order to have minimum distortion of the temperature distribution by the test huts, only the observations furthest removed from the huts were used in this analysis. At this location, 20 feet from a service tunnel, thermocouples were installed to a depth of 10 feet. No provision was made for snow clearance. Consequently, the insulating effect of the snow cover could not be independently evaluated from these data.

DESCRIPTION OF ANALYSIS

The ground temperature observations were studied by the mathematical techniques of Fourier analysis. The principal Fourier components of these temperature variations are the diurnal and the annual waves. However, there were insufficient data to determine the diurnal component. Consequently, this description is concerned only with the annual temperature wave.

The basic assumption was that the ground temperature at a fixed depth for

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any one year could be expressed by a Fourier series of the form,

$$u(x,t) = \sum_{-\infty}^{\infty} a^n(x)\, e^{in\omega t}, \tag{1}$$

where:

$u(x,t)$ is the ground temperature at depth x and time t,
$a_n(x)$ is the amplitude of the Fourier component of frequency $n\omega$,
n is zero or an integer, and
ω is the fundamental frequency, 2π radians per year.

The functions $a_n(x)$ are given by the following integral,

$$a_n(x) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} u(x,v)\, e^{-in\omega v}\, dv. \tag{2}$$

These integrals were determined graphically from the observations. The precision of the data was such that reliable results could only be obtained for $n = 0, \pm 1$, which are the components of the annual wave.

This development neglects certain transient features of the ground temperature variation. This was estimated to be a small effect, however, because the amplitude of the annual temperature wave only changes by a small amount from one year to the next.

The behaviour of the annual temperature wave was then compared with the theory of steady state heat conduction in a homogeneous semi-infinite solid subject to a harmonic surface temperature. The theoretical expression for the temperature at depth x and time t is:

$$u(x,t) = A_0\, e^{-\left(\sqrt{\frac{\omega}{2k}}\right)x + i\left[\omega t - \delta - \left(\sqrt{\frac{\omega}{2k}}\right)x\right]} \tag{3}$$

where,

A_0 is the amplitude of the surface temperature wave,
δ is the initial phase angle of the surface temperature wave,
ω is the frequency of the temperature wave, and
k is the thermal diffusivity of the ground.

Equation (3) predicts an increasing phase lag and a decreasing amplitude with depth for any temperature wave. Furthermore, the magnitude of both the attenuation and the phase shift is directly proportional to the square root of the quotient of the frequency by the thermal diffusivity. Plots of the phase angle and the logarithm of the amplitude against depth for the field observations yield a check on the applicability of the theory. If this theory gives an adequate description of the behaviour of the annual temperature wave, such curves should be linear in x with a slope of magnitude $\sqrt{\omega/2k}$.

Results of the Analysis

The results indicate that the annual temperature variations at the test sites are consistent with a homogeneous semi-infinite solid model for the soil. Figures 1 and 2 are examples of phase and amplitude plots, illustrating the linear behaviour with approximately equal slopes. It also was observed that the thermal diffusivity of each

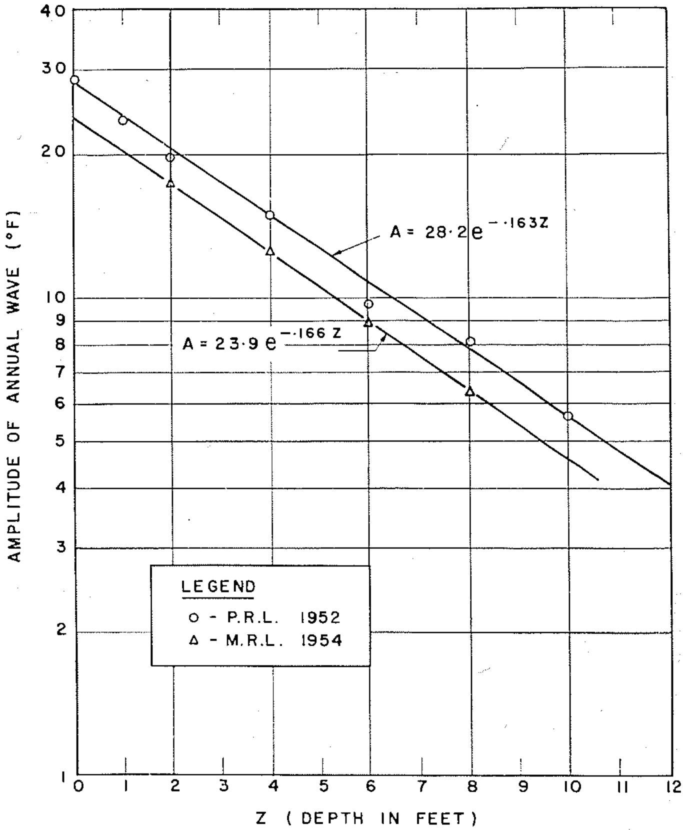

Fig. 1 — Annual wave amplitude as a function of depth.

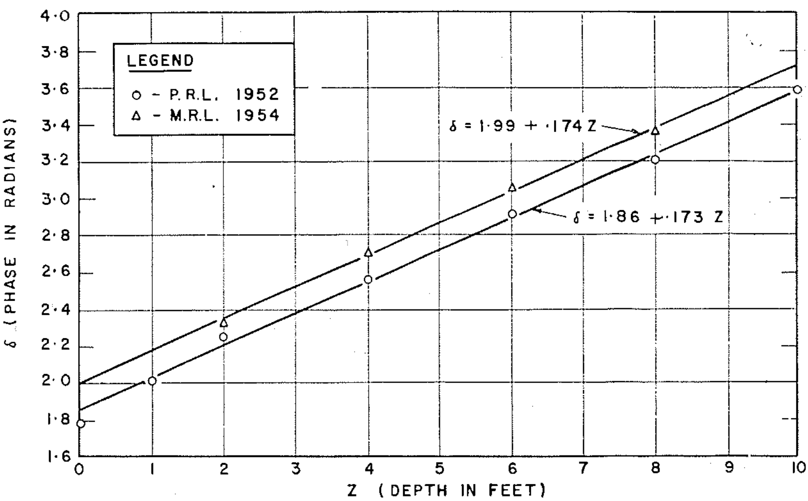

Fig. 2 — Phase of annual wave as a function of depth.

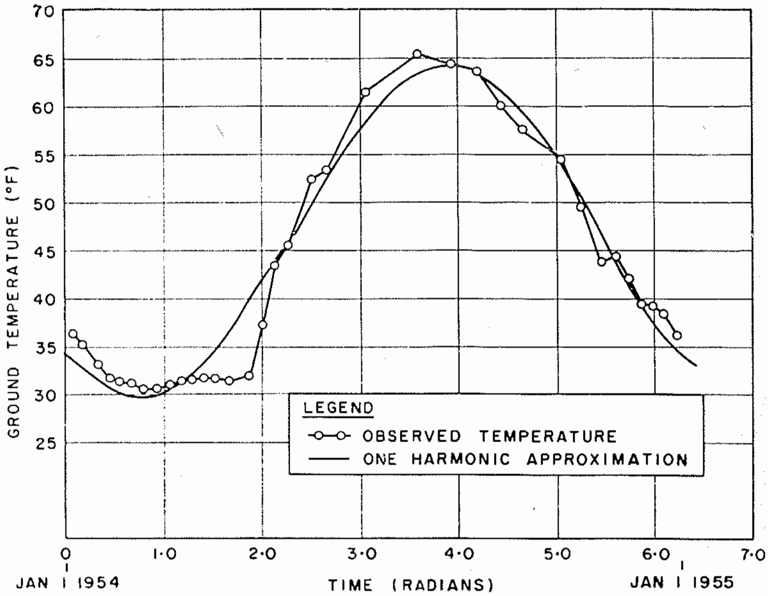

Fig. 3 — One harmonic ground temperature approximation (M. R. L. — 1954 Depth — 2 ft).

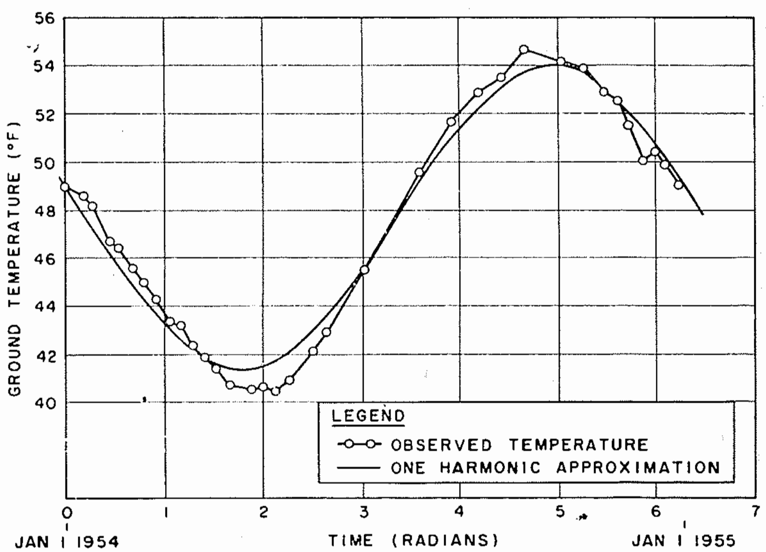

Fig. 4 — One harmonic ground temperature approximation (M. R. L. — 1954 Depth — 8 ft).

soil type considered did not vary appreciably from year to year. The value of the constant $\sqrt{\omega/2k}$ ranged from 0.13 ft.$^{-1}$ for sand to 0.17 ft.$^{-1}$ for clay. The annual wave is a good approximation to the actual ground temperature. Figures 3 and 4 give one comparison of this first harmonic approximation with the observed temperature at two different depths for the same test pit. This illustrates that the approximation improves with increasing depth. This is to be expected since the higher harmonics of the surface temperature are attenuated more rapidly with depth.

This first harmonic approximation yields a simple formula for the maximum penetration of any isotherm. The equation is:

$$x_m = \sqrt{\frac{2k}{\omega}} \log_e \left(\frac{A_0}{u_0 - u_i}\right), \tag{4}$$

where,

x_m is the maximum penetration of the isotherm u_i, and
u_0 is the mean ground temperature.

If the depth of freezing is defined as the maximum penetration of the 32° isotherm, this depth can be determined by setting $u_i = 32°$ in equation (4). The observations indicate that u_0 is approximately independent of x so equation (4) is readily soluble. Graphical solutions can be obtained if u_0 is a function of the depth.

The observations also showed the quantities $\sqrt{\omega/2k}$ and u_0 to be nearly constant from year to year. Thus the change in depth of penetration of any isotherm from year to year can be calculated from the following expression:

$$x_{1i} - x_{2i} = \sqrt{\frac{2k}{\omega}} \log_e \left(\frac{A_{10}}{A_{20}}\right) \tag{5}$$

where,

x_{1i} is the depth of penetration of the isotherm u_i for the year when the surface temperature amplitude is A_{12}, and
x_{2i} is the penetration of the isotherm u_i for the year when the surface temperature amplitude is A_{20}.

It would appear that a knowledge of the surface temperature and hence A_0 is required to use equation (4). This is not true because equation (3) affords a method of deter mining A_0 if the amplitude of the temperature wave is known at one depth. Thus a knowledge of the soil thermal diffusivity and a temperature variation at one depth is sufficient to use equation (4). Equation (5) is equally general. The minimum temperature for the jth year $u_{jm}(x_0)$ at an arbitrary depth x_0, in first harmonic approximation is:

$$u_{jm}(x_0) = u_0 - A_0 e^{-\left(\sqrt{\frac{\omega}{2k}}\right)x} \tag{6}$$

Thus the following relation is true:

$$\frac{A_{10}}{A_{20}} = \frac{u_0 - u_{1m}(x_0)}{u_0 - u_{2m}(x_0)}. \tag{7}$$

Equation (7) states that the ratio of the surface temperatures for any two years can be determined from a knowledge of the amplitude of the temperature at an arbitrary depth. Thus equation (5) states that in the first harmonic approximation the difference in penetration of any isotherm from year to year can be calculated from a knowledge of the soil thermal diffusivity and a time -temperature record at one depth.

Air temperature is a commonly used index of frost penetration. The mean daily air temperature near the field installations as furnished by meteorological records was also analysed into its Fourier components. This analysis showed no simple relationship to exist between ground surface and air temperature. The amplitude of the annual component of the mean daily air temperature was greater than the annual amplitude of the ground surface while the mean annual air temperature was less than the mean annual ground temperature. Thus air temperature cannot be used directly in these formulae. However, an observed proportionality between the annual amplitudes of the ground surface and the mean daily air temperature is thought to be of some value. As a result of this proportionality, the ratio of the air temperature amplitudes for any two years can be used in place of the ratio of the surface temperature amplitudes to determine the change of isotherm penetration from equation (5).

The Effect of Variable Soil Thermal Properties

Additional calculations designed to estimate the influence of variations in soil thermal properties indicated a possible reason why the homogeneous semi-infinite model for the soil was so satisfactory. Several different variations of the thermal properties with depth were studied. The results for each analysis were quite similar. The conclusion was that the amplitude of a temperature oscillation is relatively insensitive to a change in soil thermal properties provided this change occurs within a distance that is small as compared to the length of the temperature wave. The annual temperature wave satisfies this condition so small variations of the soil thermal properties do not show up in this analysis. The calculation for one conductivity-depth dependence is given in Appendix A.

The Effect of Snow Cover

An attempt was made to include the effect of the snow cover in the theory of the annual variation of the ground temperature. Observations from the snow-covered sites indicated that a homogeneous semi-infinite model for the soil was adequate to describe the annual temperature variations. The only apparent effect of the snow cover was to reduce the amplitude of the surface temperature and thus to reduce the penetration of a given isotherm.

For this calculation it was assumed that the heat capacity of the snow was negligible and that transients associated with changes in snow depth could be neglected. Thus it is a very rough approximation to the actual situation. The main justification of this approach was that it resulted in fair agreement with the observations.

The basic idea of the analysis was to determine an attenuation coefficient characteristic of a particular winter snow cover. If the amplitude of the annual surface temperature wave without snow cover is A_0, then in the presence of snow the amplitude will be ηA_0. This defines the attenuation coefficeint η. It depends on both the depth and time of deposition of the snow cover. The method of evaluating η is described in Appendix B.

Soil temperatures in a region having snow cover can be calculated from equations (4), (5) and (6) if A_0 is replaced by ηA_0. Thus ηA_0 can be considered as an effective surface temperature amplitude. The temperature distribution beneath non-snow-covered ground then is a special case with $\eta \equiv 1$.

The difference in depth of penetration of an isotherm with and without snow cover is given by the equation:

$$\Delta x_s = x_m - {}_s x_m = \sqrt{\frac{2k}{\omega}} \log_e\left(\frac{1}{\pi} \cdot \frac{{}_s u_0 - u_i}{u_0 - u_i}\right) \tag{8}$$

where,

x_m is the maximum penetration of the isotherm u_i in the absence of a snow cover,

${}_s x_m$ is the maximum penetration of the isotherm u_i with a snow cover,

η is the attenuation constant of the snow cover,

${}_s u_0$ is the mean ground temperature under a snow cover, and

u_0 is the mean ground temperature in the absence of a snow cover.

The Ottawa observations were compared with values calculated from equation (8). Table I gives some of these results.

TABLE I

u_i(°F)	u_0	u_0	η	x_m	${}_s x_m$	x_s measured (ft.)	x_s calc. (ft.)
40	47.7	47.9	.878	7.7	6.8	0.9	0.9
35	47.7	47.9	.878	4.2	3.2	1.0	0.9
32.5	47.7	47.9	.878	2.9	1.2	1.7	0.8

The greatest deviation between the theory and the observations occurs for isotherms located near the ground surface. This is attributed to the contribution of the higher harmonics, which are largest in this region and which are neglected in this theory.

Conclusion

This analysis has shown that at certain test locations, the annual variation of the ground temperature is closely approximated by the theory of sinusoidal temperature variations in a homogeneous semi-infinite solid. As a consequence of the attenuation term, the approximation improves with increasing depth. Further analysis has demonstrated that the amplitude of the annual temperature oscillation appears to be relatively insensitive to variations in soil thermal conductivity. This fact probably accounts for the agreement of the observations with a theory containing only one parameter — the average soil thermal diffusivity. For the soils considered, the latent heat associated with freezing and thawing appeared to have little influence on the annual amplitude of the ground temperature oscillation. Consequently, it has been possible to develop a formula that yields maximum penetration of any isotherm. This includes the depth of frost penetration if the freezing plane is identified with 32° isotherm. These calculations require only a knowledge of the soil thermal diffusivity and a temperature record at one fixed depth. It also has been possible to include an approximation for the insulating effect of the snow cover in this formulation.

It has been a practice in predicting frost penetration to use air temperatures in place of surface temperatures. The relationship between these two temperatures has been observed to be complex so the interchangeable use of these temperatures is questionable. However, a proportionality between the annual amplitude of the air and surface temperatures was observed to exist. This was true with or without snow cover. Consequently, year-to-year changes in the maximum depth of penetration of an isotherm could be determined from air temperature amplitudes. A greater understanding of the mechanism of heat exchange at the ground surface is required. however, to predict ground temperatures reliably from meteorologic variables.

The foregoing conclusions only apply to temperature distributions that vary slowly with time. An underlying hypothesis of the development was that heat flow in the ground obeys simple conduction theory. This assumption is not strictly accurate, however, due to the presence of water in the soil. It is quite possible that the average thermal diffusivity of the soil will depend on the time rate of change of temperature. Consequently the extrapolation of the results for the annual wave to other frequencies is not justified. A study of temperature distributions that vary more rapidly with time is required for a complete understanding of the ground temperature regime.

This study indicates that ground temperature variations of long period can be described by simple heat conduction theory. This implies that theory can be used to obtain a useful approximation to the solution of practical problems involving temperature fields that vary slowly with time. Furthermore, appropriate values for the thermal parameters for several locations in Canada are available from the field observations.

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APPENDIX A

Steady state heat concuction in l.c. semi-infinite solid with non-constant thermal properties

The purpose of this analysis was to estimate the influence of variations in the thermal conductivity on the amplitude of the temperature waves. The solution described here is for only one of several assumed conductivity-depth variations. The results for each variation were quite similar so this particular solution was assumed to contain the important features of the system. This solution is the simplest of those attempted.

Analysis of the field observations showed that the greatest deviations from the simple theory occurred near the ground surface. It was therefore concluded that the greatest variation in the thermal conductivity would also be near the surface. This fact was incorporated into the calculation by assuming that the conductivity was constant for depths greater than a fixed depth L. Furthermore, the specific heat of the soil was assumed to be constant. The steady state problem, subject to a sinusoidal surface temperature, can then be formulated as follows:

$$\frac{d}{dx}\left(K_0 g(x)\frac{df(x)}{dx}\right) - i\,\omega\,\rho\,c\,f(x) = 0,$$

where,

$K = K_0 g(x)$	is the soil thermal conductivity,
$f(x)e^{i\omega t}$	is the temperature at depth x and time t,
c	is the specific heat of the soil, and
ρ	is the density of the soil.

For this particular solution $g(x)$ was assumed to have the following form:

$$g(x) = \begin{cases} \alpha^2; & 0 < x < L \\ 1; & L < x \end{cases}$$

where,

α^2 is a constant not equal to unity.

Introduce the transformation

$$u = (1 - x/L).$$

Thus the equations become,

$$\frac{d}{du}\left(g(u)\frac{df(u)}{du}\right) = \frac{i\omega\rho c L^2}{K_0} f(u) = \beta^2 f(u)$$

$$g(u) = \begin{cases} \alpha^2; & 1 > u > 0 \\ 1; & 0 > u \end{cases} \tag{1a}$$

The solution of the system of equation (1a) is,

$$1 > u > 0; \qquad u = Ae^{\frac{\beta u}{\alpha}} + Be^{\frac{-\beta u}{\alpha}} \tag{2a}$$

$$0 > u; \qquad u = Ce^{\beta u} \tag{3a}$$

Assume the constant C to be fixed so that the constants A and B can be determined from the continuity conditions at $u = 0$. These conditions give:

$$C = A + B \tag{4a}$$

$$C = \alpha A - \alpha B \tag{5a}$$

Thus the solutions are

$$1 > u > 0; \quad u = \frac{C}{2\alpha}\left[(\alpha + 1)\, e^{\frac{\beta u}{\alpha}} + (\alpha - 1)\, e^{\frac{-\beta u}{\alpha}}\right] \tag{6a}$$

$$0 > u; \qquad u = Ce^{\beta u} \tag{7a}$$

For a constant thermal conductivity the value of the temperature amplitude at the surface is given by

$$A_1 = |Ce^{\beta}| \tag{8a}$$

For the selected variation of the conductivity, the surface amplitude is given by

$$A_2 = \left|\frac{C}{2\alpha}\left[(\alpha + 1)\, e^{\beta/\alpha} + (\alpha - 1)\, e^{-\beta/\alpha}\right]\right| \tag{9a}$$

The difference between A_1 and A_2 is a measure of the change of the temperature distribution caused by the non-constant thermal conductivity.

Select the phase so that C is a real number. Then

$$A_1 = Ce^{\theta} \; ; \; \theta = \left(\sqrt{\frac{\omega\rho c}{2K_0}}\right) L \tag{10a}$$

and

$$A_2 = \frac{(\alpha + 1)}{2\alpha} Ce^{\theta/\alpha} \;\; (1 + 2\gamma \cos\frac{2\theta}{\alpha} + \gamma^2)^{1/2} \tag{11a}$$

where

$$\gamma = \left(\frac{\alpha - 1}{\alpha + 1}\right) e^{-\frac{2\theta}{\alpha}}$$

Thus for a fractional change in K, given by $y(\alpha) = (\alpha^2 - 1)$, the corresponding fractional change in the surface amplitudes is given by

$$\chi(\alpha,\mathrm{L}) = \frac{\mathrm{A}_1 - \mathrm{A}_2}{\mathrm{A}_1} = 1 - \frac{(\alpha + 1)}{2\alpha}\, e^{\left(\frac{1}{\alpha} - 1\right)\theta} \left(1 + 2\gamma \cos\frac{2\theta}{\alpha} + \gamma^2\right)^{1/2} \tag{12a}$$

If the values,

$\theta/\mathrm{L} = 0.17$ ft.$^{-1}$ (observed value for clay)

and $\mathrm{L} = 2$ ft.

are used, the following results are obtained

$y(\alpha)$	$x(\alpha,\mathrm{L})$
25%	1.4%
100%	16%

Thus, for the annual wave, significant changes in the conductivity are required to produce measurable changes in the surface amplitude. However, it should be noted that $x(\alpha,\mathrm{L})$ increases both with frequency and with L. In general, a larger value of the ratio of the distance L to the length of the temperature wave will result in a larger perturbation of the temperature amplitude.

Appendix B

Harmonic temperature variations in snow-covered ground

One of the assumptions on which this analysis is based is that the heat storage in the snow can be neglected compared to the heat storage in the ground. Thus the differential equation of heat flow in the snow is,

$$\frac{\partial^2 u(x)}{\partial x^2} = 0 \tag{1b}$$

with boundary conditions

$$u(0) = u_i \;;\; u(-l) = u_0 \tag{2b}$$

$u(x)$ is the temperature in the snow cover
$x = 0$ is the ground snow interface
$x = -l$ is the upper surface of the snow.

The solution of equations (1b) and (2b) is

$$u(x) = (u_i - u_0)\, x/l + u_i$$

Thus heat flow through the snow is

$$\mathrm{G} = -\mathrm{K}_s \frac{du(x)}{dx} = \mathrm{K}_s \frac{(u_0 - u_i)}{l}$$

where K_s is the thermal conductivity of the snow. Thus, to this approximation, the snow behaves as a film with a film coefficient $h_s = \mathrm{K}_s/l$. This coefficient is time dependent because the snow depth «l» varies with the time.

The boundary value problem of heat flow in the ground, subject to this surface film condition is formulated as follows:

$$\frac{\partial^2 v(x,v)}{\partial x^2} = \frac{\rho c}{K_g}\frac{\partial v}{\partial t} \quad (x,t)\ 0 \leqslant x < \infty \tag{3b}$$

$$\lim_{x \to \infty} v(x,t) = 0 \tag{4b}$$

$$-K_g \frac{\partial x}{\partial v}(0,t) = h_s(t)\,[u_0 - v(0,t)] \tag{5b}$$

where

K_g is the thermal conductivity of the ground

$K_g/\rho c$ is the thermal diffusivity of the ground.

This is an extremely complicated system of equations due to the time dependence of the coefficient $h_s(t)$. To carry the analysis further, it was necessary to resort to crude approximations whose major justification was in the agreement that was obtained with the field observations.

The approximation consists of a complete neglect of the thermal transients associated with changes in the snow depth. Thus, a change in $h_s(t)$ will result in an instantaneous change in $v(x,t)$. Consequently h_s may be treated as a constant so the boundary value problem becomes quite simple. For the steady state periodic solution

$v(x,t) = W(x)\,e^{i\omega t}$ with a surface temperature

$v(0,t) = A_0\,e^{i\omega t}$ the equations become,

$$\frac{d^2W(x)}{dx^2} - \frac{i\omega\rho c}{K_g}W(x) = 0 \tag{6b}$$

$$-\frac{dW(0)}{dx} = \frac{h_s}{K_q}(A_0 - W(0)) \tag{7b}$$

The solution of this system of equations is,

$$v(x,t) = \frac{A_0\, e^{i\omega t - \left(\sqrt{\frac{i\omega\rho c}{K_g}}\right)x}}{\left(1 + \frac{K_g}{h_s}\sqrt{\frac{i\omega\rho c}{K_g}}\right)} \tag{8b}$$

Thus the temperature at the ground snow interface can be written

$$v(0,t) = \frac{A_0}{\sqrt{1 + 2\theta + 2\theta^2}}\, e^{i\left(\omega t - \tan^{-1}\left(\frac{\theta}{H\theta}\right)\right)} \tag{9b}$$

$$\text{with} \quad \theta = \frac{K_g}{h_s}\sqrt{\frac{\omega\rho c_e}{2K_g}}$$

Finally, the phase shift resulting from the snow was neglected. Thus, to this approximation the ground surface temperature is given by the equation,

$$v(0,t) = \frac{A_0}{\sqrt{1 + 2\theta + 2\theta^2}}\, e^{i\omega t} \tag{10b}$$

Equation (10b) can be rewritten,

$$v(0,t) = \zeta(t)\, A_0\, e^{i\omega t} \tag{11b}$$

where $\zeta(t)$ is an attenuation function for the snow cover given by the relation

$$\zeta(t) = (1 + 2\theta + 2\theta^2)^{-1/2}.$$

$\zeta(t)$ can be calculated from a knowledge of the snow depth and the thermal properties of the soil and snow.

Equation (11b) gives an effective surface temperature to be used in the theory of heat flow in a semi-infinite solid. The attenuation constant, η defined previously can be calculated from $\zeta(t)$. A description of this calculation follows. Assume the annual component of the surface temperature is given by the equation

$$u(t) = A_0 \sin \omega t + b_0 \cos \omega t \tag{12)b}$$

In the absence of a snow cover, $u(t)$ is what has been previously called the ground surface temperature. However, with snow cover, the ground snow interface temperature is given by,

$$u_s(t) = \zeta(t)\, a_0 \sin \omega t + \zeta(t)\, b_0 \cos \omega t \tag{13b}$$

The function described by equation (13b) is not a simple sine wave. The annual component of $u\ (t)$, $u_{sa}(t)$ was then determined. This annual component is then represented by the equation:

$$u_{sa}\ (t) = a_1 \sin \omega t + b_1 \cos \omega t \tag{14b}$$

The constants a_1 and b_1 are given by the following expressions:

$$a_1 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} (a_0 \sin \omega\, t + b_0 \cos \omega\, t)\, \zeta(t) \sin \omega\, t\, dt$$

$$b_1 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} (a_0 \sin \omega\, t + b_0 \cos \omega t)\, \zeta(t) \cos \omega\, t\, dt$$

The constants a_1 and b_1 were determined by numerical integration.

The attenuation coefficient η can be determined from equations (12b) and (14b) It is given by the expression:

$$\eta = \sqrt{\frac{a_1^2 + b_1^2}{a_0^2 + b_0^2}} \tag{15b}$$

The values of η presented in Table I were determined by such a calculation. The value of the thermal conductivity of the snow was obtained from the work of Yosida et al*. A snow density of 0.28 gm/cm³ was assumed to be an average value for settled snow. The thermal diffusivity of the soil was obtained from the ground temperature analysis. The heat capacity of the soil was calculated by the method suggested by Kersten** using field measurements of the soil density and the soil water content. Finally the soil thermal conductivity was calculated from the heat capacity and thermal diffusivity.

* Yosida, Z. and H. Iwai. Measurement of the thermal conductivity of snow cover. Snow, Ice and Permafrost Research Establishment, *Translation* No. 30. Nov. 1954.

** Kersten, M. S. Thermal properties of soils. Highway Research Board, Special Report No. 2, 1952. p. 161.

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