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by S.B. Beale

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A DIRECT METHOD FOR GRID GENERATION

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ABSTRACT

A method to generate body-fitted grids, based on the direct solution for three scalar functions, is derived. The grid is re-meshed, based on the solution, using a grid-correction procedure. Calculations are performed for a variety of problems with both Dirichlet and Neumann boundary conditions, involving the use of non-linear source-terms, and other techniques to effect grid-control.

1. INTRODUCTION

Numerical methods are readily applied to a wide-range of problems in fluid flow, heat, and mass and mass transfer, governed by the single equation,

$$D(\phi) = \underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{Transient}} + \underbrace{\vec{\nabla}\cdot(\rho\vec{u}\phi)}_{\text{Convection}} - \underbrace{\vec{\nabla}\cdot\Gamma\vec{\nabla}\phi}_{\text{Diffusion}} - \underbrace{S}_{\text{Source}} = 0 \tag{1}$$

Many problems have been successfully solved using computational fluid dynamics (CFD) codes, based on a finite-volume method [1,2] where $\phi = u, v, w, h, \ldots$ are solved iteratively. In this paper it is shown how the grid variables may be added to the list. Although grid smoothing and control are an integral part of the overall solution procedure, they are generally treated as a separate subject. Books [3,4] and review articles [5,6,7] have appeared on the subject.

Grid-generation involves the stipulative definition of functions, denoted by Greek letters, ξ^i (or alternatively $\xi,\ \eta,\ \zeta$) using differential equations,

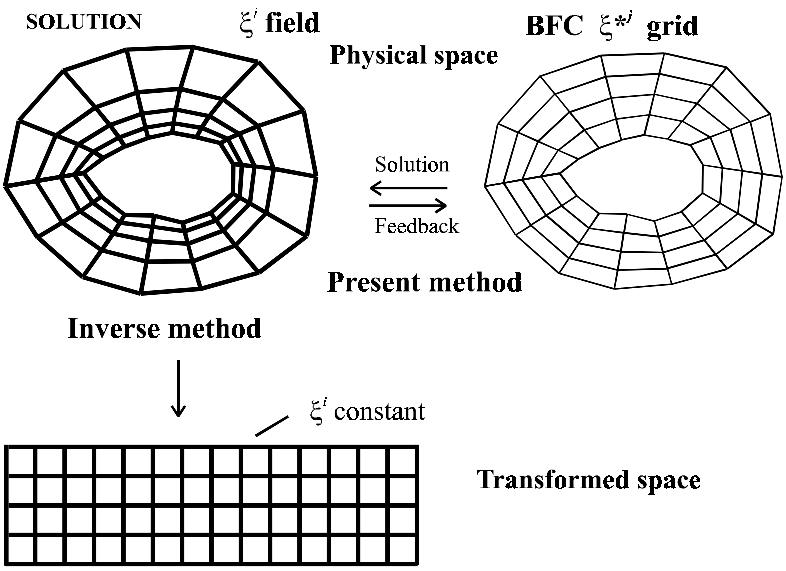


Figure 1. Conceptual schematic of methodology

$$\frac{D\left(\xi^i\right)}{D\left(x^j\right)} = 0 \tag{2}$$

where English letters, x^j (or x, y, z), denote Cartesian components. Equation (2) is sometimes referred to ostensively as a 'physical-space' formulation. In grid generation, inverse methods [8] are often employed, namely,

$$\frac{D\left(x^i\right)}{D\left(\xi^j\right)} = 0 \tag{3}$$

i.e., the Cartesian co-ordinates of the grid are obtained in 'transformed space'. Most schemes are based on inversions of (2), however transformed-space formulations have also been proposed [9,10]. These caused grid-folding, though more recent forms [11,12] are in use. In this paper, a co-ordinate independent formulation using vector operators is made. Thus any co-ordinate system may be employed to obtain a solution,

$$D\left(\xi^i\right) = \frac{D\left(\xi^i\right)}{D\left(\xi^{*j}\right)} = 0 \tag{4}$$

ξ^{*j} could represent a Cartesian, polar, or general BFC system (grid), as is used here: An initial ξ^{*j} grid is generated algebraically, and the equations for $\phi = \xi^i$ discretized in physical space. The key to the procedure is that values of the solved-for ξ^i scalars are back-substituted into the grid, as the solution proceeds.

The governing equations may be parabolic, hyperbolic or elliptic, the latter being popular. First-used elliptic equations were Laplace systems [13] where only the diffusion term in (1) is non-zero. These systems satisfy an extremum principle, namely that the mapping be proper; 1-1 and monatonic, boundary-values spanning the interior. Diffusion-source equations [14,15] are also widely used. ‘Control-functions’ are often coded as source terms, in order to alter the position or slope of grid-lines/surfaces. Thompson et al. [16] proposed exponential terms to effect attractions. These can violate the extremum principle, and for this and other reasons, these functions have been somewhat supplanted by more recent rationales based on boundary orthogonality [17,18]. Both approaches are considered.

2. FINITE-VOLUME EQUATIONS

The so-called ‘mathematical’ form [19] of Eq. (1) is,

$$D(\phi)=\frac{\partial}{\partial t}\left(\sqrt{g}\phi\right)+\frac{\partial}{\partial\xi^{*j}}\left(\sqrt{g}u^j\phi-\sqrt{g}g^{jk}\frac{\partial\phi}{\partial\xi^{*k}}\right)-\sqrt{g}S=0 \qquad (5)$$

where $\phi=\xi^i=\xi,\eta,\zeta$ for i = 1,2,3. The metric components, g^{jk}, and Jacobian, $\sqrt{g}$, refer to the ξ^{*i} co-ordinate system. Only restricted forms, such as diffusion-source systems, will be considered below in any detail. Equation (5), or equivalent physical form may be discretized as [20],

$$\phi_P=\frac{a_W\phi_W+a_E\phi_E+a_S\phi_S+a_N\phi_N+a_L\phi_L+a_H\phi_H+a_T\phi_T+CV}{a_W+a_E+a_S+a_N+a_L+a_H+a_T+C} \qquad (6)$$

where W, E, S, N, L, H refer to the West (i-1), East (i+1), South (j-1), North (j+1), Low (k-1), High (k+1) neighbours of P. T-values refer to the previous grid (inertial relaxation).

2.1 Grid correction

Suppose a BFC grid, with $\vec{r}_P*=\left(x_P*,y_P*,z_P*\right)$, has been generated. Let ξ_{ref}, η_{ref}, ζ_{ref} be desired reference values: These are often natural numbers, but could be real numbers or values of ξ_P, η_P, ζ_P at particular nodes. If the grid corners are not at the desired places, ξ_P, η_P, ζ_P will differ from ξ_{ref}, η_{ref}, ζ_{ref}. Displacement correction factors $\vec{r}\,'$ are then added as follows,

$$\vec{r}_p=\vec{r}_P*+\alpha\vec{r}_P' \qquad (7)$$

$$\vec{r}_P' = \left(\xi_{ref} - \xi_P\right)\frac{\partial \vec{r}}{\partial \xi} + \left(\eta_{ref} - \eta_P\right)\frac{\partial \vec{r}}{\partial \eta} + \left(\zeta_{ref} - \zeta_P\right)\frac{\partial \vec{r}}{\partial \zeta} \qquad (8)$$

α is a linear relaxation coefficient, and,

$$\left(\frac{\partial \vec{r}}{\partial \xi}, \frac{\partial \vec{r}}{\partial \xi}, \frac{\partial \vec{r}}{\partial \xi}\right) = \left(\frac{\vec{r}_E - \vec{r}_W}{\xi_E - \xi_W}, \frac{\vec{r}_N - \vec{r}_S}{\eta_N - \eta_S}, \frac{\vec{r}_H - \vec{r}_L}{\zeta_H - \zeta_L}\right) \qquad (9)$$

Often one may assume, $\xi_E - \xi_W = 2$, etc. Scalar fields are re-initialised,

$$\phi = (1 - \alpha)\phi_P{}^* + \alpha\phi_{ref}, \qquad \phi = \xi, \eta, \zeta \qquad (10)$$

and the process repeated. The numerical solution of (1) together with (7) constitutes a complete description of the methodology.

<u>2.2 Boundary conditions</u>

Boundary conditions are prescribed as linearized source terms, as follows,

$$S = C\left(V - \phi_P\right) \qquad (11)$$

(i) Neumann problem: Normal gradients $\partial\phi/\partial n$ are equivalent to fixed sources. $\partial\phi/\partial n = 0$ corresponds to the default $S = 0$. (ii) Dirichlet problem*:* ϕ-values are fixed to V with a large coefficient, C. V should be consistent at opposing nodes in 2D or around an entire ‘slab’ in 3D. (iii) Mixed Dirichlet/Neumann problem: corresponding to the ‘natural’ boundary value problem, will produce good results under most circumstances. Two fixed-value Dirichlet boundary conditions are required in addition to two (2D) or four (3D) Neumann boundary conditions. Practical considerations often prevent this formulation. Other Dirichlet/Neumann combinations are encountered: e.g., fixed values along the 4 bounding lines/curves for the Neumann surfaces in case (iii), above.

Grid correction is not necessary at Dirichlet boundaries. At Neumann boundaries, grid correction is applied subject to an additional constraint, e.g., $\xi(x, y, z) =$ constant; unless boundary orthogonality is induced using variable source-terms, in which case grid correction should not be applied.

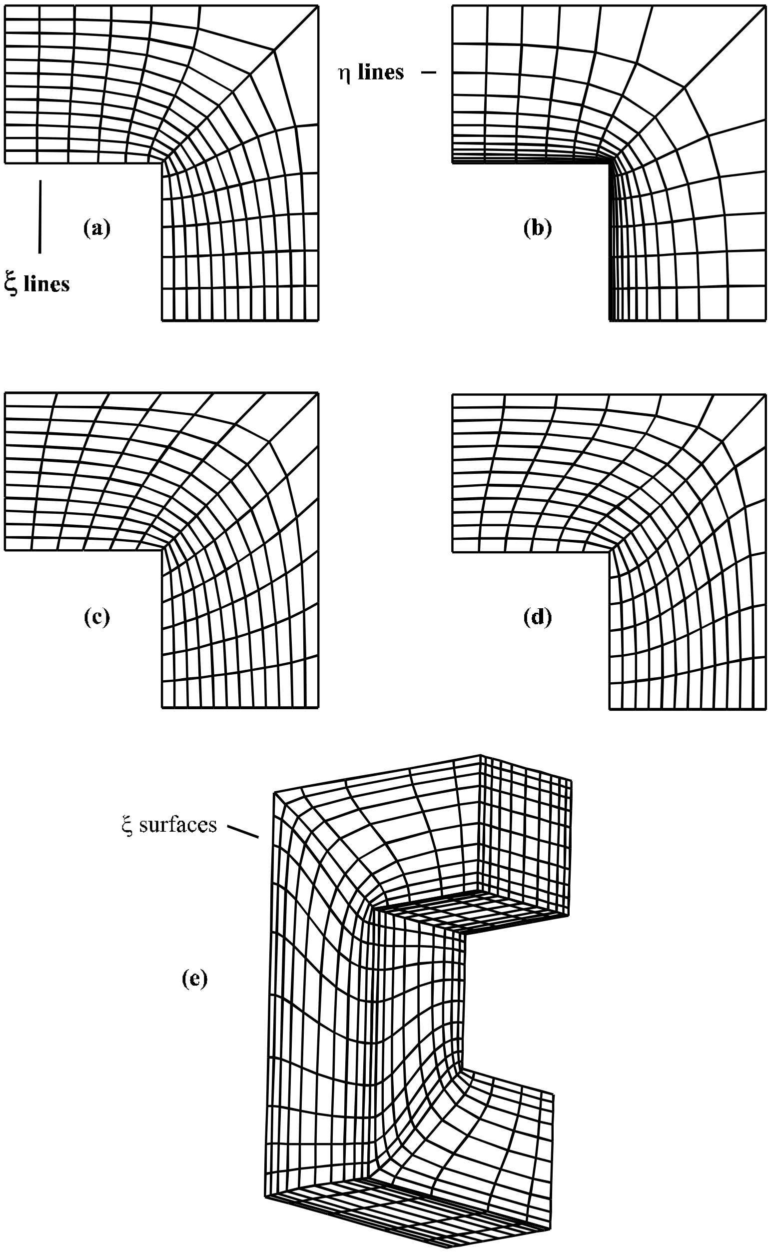


Figure 2. Grid generation examples

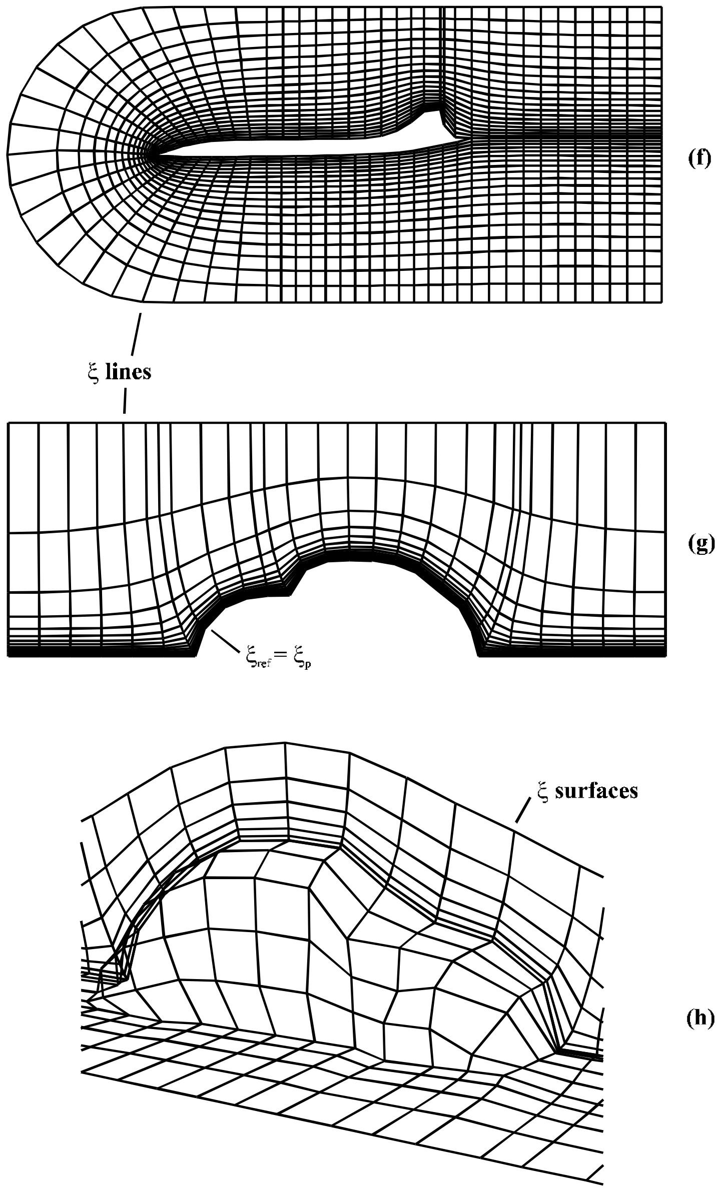
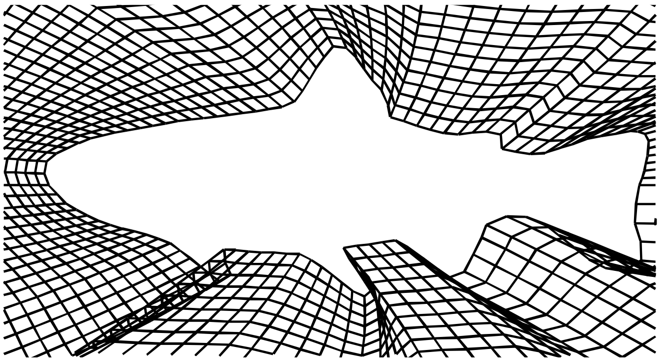
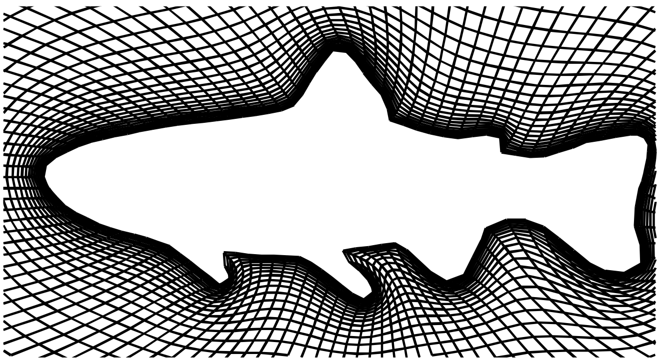


Figure 2. (continued)



(i)



(j)

Figure 2. (continued)

3. EXAMPLES

Figure 2(a)-(d) show four elementary O-grids. Figure 2(a) is a Laplace system corresponding to the mixed boundary-value problem with $\Delta\xi_{ref} = \Delta\eta_{ref} = 1$. In Fig. 2(b) η cells have concentrated. This is achieved by one of two equivalent methods (a) by introducing a non-linear source term, S, for example as,

$$S = \left(\frac{n-1}{L^2}\right) \times \frac{\left(e^p - 1\right)^2}{p} \exp\left(-2p\frac{\eta_P - 1}{n-1}\right) \qquad (12)$$

where L is a length scale, $n-1$ is the number of cells in the η-direction, and p controls the distribution. (b) By prescribing η_{ref} as a set of non-integer values. Figure 2(c) shows a Dirichlet-Laplace system for $\Delta\xi_{ref} = \Delta\eta_{ref} = 1$. Figure 2(d) demonstrates the use of variable source terms to obtain boundary orthogonality. These were prescribed iteratively as,

$$S = S^* + S' \tag{13}$$

S^* is a previous value and,

$$S' = C\left(\xi_{ref} - \phi_P\right), \quad \xi_{ref} = 1,2,3,\ldots n \tag{14}$$

and $C = a_E + a_W$. Interior source terms were interpolated from boundary-sources, weighted according to the Jacobian, $\sqrt{g}$. The same end may be also achieved by varying reference values. Figure 2(e) shows a 3D bend, where variable source terms are prescribed, for the latter case in §2.2, namely fixed values along the 4 bounding lines of each Neumann surface. Figure 2(f) shows a C-grid around an aircraft: Only η was solved as a Dirichlet problem, ξ lines are algebraically-generated initial values. Figure 2(g) shows an H-grid over a 2D car body. The grid was allowed to slide at the upper boundary, but fixed at the lower wall; ξ_{ref} values were set to ξ_P at $j = 1$. Figure 2(h) is a similar grid in 3D. In Fig. 2(i)(j), an initially-folded O-grid has been unfolded and concentrated using the above procedure.

4. DISCUSSION

The results demonstrate grid generation by means of the solution of Eq. (1) using the correction procedure Eq. (7). The main differences between this method and a conventional procedure are: (a) The dependent variables are ξ, η, ζ not *x*, *y*, *z*. (b) Diffusion-coefficients are based on a conservative discretization; $\left(\sqrt{g}g^{11}\right)_w$ $\left(\sqrt{g}g^{11}\right)_e$ at (*i*-½,*j*) (*i*+½, *j*), not node-centred values g_P^{ii} (the same is true for the non-orthogonal terms). (c) Source-terms are just $\sqrt{g}S_P$; no cross-terms, e.g., η source in ξ equation etc., are required.

The sign of *p* in Eq. (12), determines the boundary to which grid lines are attracted. The strategy was derived from consideration of the stretching-function,

$$\frac{x - x_{\min}}{x_{\max} - x_{\min}} = \frac{1 - \exp\left(p\dfrac{\xi - 1}{n - 1}\right)}{1 - \exp(p)} \tag{15}$$

A diffusion-source equation, with *S* prescribed according to (12) satisfies the inverse (logarithmic) function in 1D. The length-scale, $L = \left|\vec{r}_{\max} - \vec{r}_{\min}\right|$ should be a maximum, to assure compliance with the extremum principle. Other source terms may readily be used [21].

There is no requirement control-functions be coded as diffusion-source equations; alternative forms exist e.g., variable Γ diffusion equations [22], convection-diffusion formulation [23] and so forth. The control function in Fig. 2(g) was actually coded using a convection-diffusion formulation, and formulations such as $u = u^* + u'$ successfully used in place of Eq. (13). An alternative is to prescribe ξ_{ref} according to a bunching law, or use nodal values, instead of integers. Under these circumstances, boundary ξ-values are not fixed in the linear algebraic equations. If the same ξ_{ref} is used at opposite boundaries (2D) or round and entire slab (3D) the grid will be parallel to scalar field. Even if ξ, η, ζ are not parallel to ξ_{ref}, η_{ref}, ζ_{ref}, the same results as using variable source-terms, are effectively obtained in a rapid and stable manner.

The solution of mixed boundary-value problems with sliding boundaries is repeatable, grid-independent, and allows for effective grid-control. ξ, η, ζ are solved-for and may be controlled, independently, and stability is seldom a problem. With Neumann conditions, boundary (x,y,z) co-ordinates slide subject to, say, the constraint, $\xi(x,y,z) = \text{constant}$, by locating the point on the ξ-surface a minimum from (x^*+x', y^*+y', z^*+z). No distinction need be made between surface and regular grid generation in Euclidean space; however, a general procedure for complex shapes is not trivial and there may be constraints for the grid to pass through specific points. There will always be situations where the user is obliged to implement fixed boundary nodes. Initial grids may then be highly distorted, due to the combination of grid-bunching and trans-finite interpolation, and stability a matter for concern. When solving Dirichlet problems, Fig. 2(f), the choice of boundary point distribution and control-functions must be made consistently, e.g., Eqs. (12) and (15), or divergence will occur near the boundary and control lost. For Fig. 2(f), only η was solved; ξ values were generated using trans-finite interpolation; A useful feature of this method is that each variable may be treated independently. In many problems it is difficult to prescribe boundary points consistent with the natural solution, a priori. Another feature of the code is the facility to obtain results for ξ, η, ζ in a fixed grid ξ^*, η^*, ζ^*, by setting $\alpha = 0$ in Eq. (7), thus providing an idea of where boundary points should be located.

When using variable source-terms to produce boundary orthogonality, Eq. (13); the equations were solved as mixed (Neumann) problems, with S eliminating the discrepancy between Neumann and Dirichlet solutions. Interior sources were obtained by interpolation. Technically the ξ distribution is a function of changes in η and ζ, however, the effect is minor; only interior weightings, not end-values of S, are affected.

Care was taken to enforce the extremum principle, and grid folding [9,10] was not a problem, even when highly concentrated grids were produced, see Fig. 2(g)(h). Initially-folded grids were also unfolded, Fig. 2(i)(j) by imposing limits $1 \le \xi_P \le n$ on field variables, and ensuring $\sqrt{g}$ was always positive in the scheme.

Tests showed the procedure to be comparable in speed to inverse methods. Point-by-point, line, and slab and whole-field procedures [1] were considered. The latter accelerated convergence substantially, however point-by-point schemes were, of necessity used with Eq. (13). The code requires additional memory for ξ, η, ζ, (only one of which is required at any given time). Storage for x_P, y_P, z_P, metric-coefficients, etc., are required here as elsewhere. The current implementation was node-based, but may readily be adapted for cell-centred procedures: Because the same algorithm as the flow solver is used, the same software may be used. Available CFD codes with specific features may be exploited: e.g., multi-block and fine-grid-embedding, multi-grid acceleration, and built-in memory-management techniques. Modification of existing codes to include grid correction is a relatively simple task, and many important problems in grid generation; user-interface, domain decomposition, boundary condition prescription, and automation, are common to CFD flow solvers. Modification of the method for solution-based grid-adaptation using redistribution is apparent; since the scheme is inherently adaptive.

5. CONCLUSIONS

In previous grid generation methods, the governing equations are usually formulated in physical space, inverted, and solved using a finite-difference approximation on a uniform mesh in transformed space. In the method here, both the formulation and the solution occur in physical space.

A scheme based on the solution of the scalar transport equation was described. It was shown that by implementing a grid-correction or re-mesh scheme, grids could be generated using a conventional finite-volume type formulation. Both fixed and sliding conditions for the boundary grid points were considered. Exponential functions and variable functions designed to procure boundary orthogonality were introduced as source terms. Use of the latter slowed down convergence, somewhat. Effective grid control was also facilitated using non-integer reference values.

The governing equations need not be of the form Eq. (1); any suitable partial differential equations may be adopted, parabolic, hyperbolic or elliptic, regardless of whether they may be inverted analytically. All non-linear source-terms degrade the performance of linear equation solvers, and alternative formulations should be considered in the future. Existing CFD codes may readily be modified to do grid generation with the addition of a grid-correction scheme.

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