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Résumé

Une nouvelle approche est proposée pour tenir compte de l'aspect «infinité» des milieux non bornés analysés au moyen de la méthode des éléments finis dans le domaine temps. Le milieu est divisé en une région finie, qui présente un intérêt dans l'analyse, et un milieu résiduel non borné. L'apport en rigidité de ce dernier est bien représenté par la matrice d'influence de sa limite avec la région finie; la matrice est calculée directement dans le domaine temporel. Cette approche convient aux situations où il faut effectuer de nombreuses analyses de la région finie.

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ANALYSIS OF WAVE PROPAGATION IN UNBOUNDED MEDIA

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Abstract—A new procedure is presented to account for the radiation condition of unbounded media analysed using the finite-element method in the time domain. The medium is divided into a finite region that is of interest in the analysis, and the remaining unbounded medium. The stiffness contribution of the latter is properly represented by its boundary influence matrix, which is directly calculated in the time domain. This procedure is suited for situations in which many analyses of the finite region must be performed.

1. INTRODUCTION

In many problems of earthquake engineering and dynamic analysis of soil–structure interaction, numerical modelling of wave propagation is necessary. Numerical models involve discretization of the continuum and if nonlinearities exist they also involve temporal discretization, i.e. the use of step-by-step time integration. The finite element method is one of the most popular discretization techniques because of its effectiveness in handling irregularities and complicated geometries. This method, however, divides the space domain into discrete elements, and can therefore only deal with finite domains with well-defined boundaries. As such boundaries do not exist naturally, waves propagating towards these boundaries will be reflected back into the model. This results in false physical behaviour of the problem. To overcome this difficulty, analysts may consider the following approaches:

(1) use of a very large finite model so that waves reflected at the model's artificial boundaries do not arrive at the area of interest within the time period over which the analysis is performed;
(2) use of large finite elements to model as large a medium as possible with a minimum number of degrees of freedom; or
(3) introduction of material damping in the finite model to dissipate reflected wave energy before it arrives at an area of interest.

The first approach may not be economical because of its high storage and computational requirements. The second approach is normally not accepted from an accuracy point of view. This is because finite elements act like 'low-pass' filters with a certain cut-off frequency that depends on the size of the element—the larger the element, the lower its cut-off frequency. The cut-off frequency of large elements necessary to produce large economical models is generally lower than the frequency range of interest. The third approach has been shown to be inappropriate [1]. These difficulties have motivated the development of so-called 'transmitting' or 'silent' boundaries. The function of these boundaries is to introduce appropriate force and/or displacement conditions to simulate the effect of the truncated exterior infinite domain and hence preserve the real physical behaviour of the problem. Many such boundaries have been developed and implemented for analyses in the time domain, with varying degrees of success. They represent only an approximation to the actual boundary condition [2]. In general, they suffer from one or more of the following drawbacks:

(1) they are effective only for a small range of incidence angles,
(2) they are not applicable to nonlinear analyses, and
(3) they may fail under static loads.

Consequently, use of a large model remains the only available approach to solve wave propagation problems directly with a high degree of accuracy in the time domain. This paper introduces an approach for the economical analysis of large models when many analyses of the same model are required. The approach is designed for explicit time integration methods. It is based on calculating a boundary influence matrix by analysing an extensive model of the exterior infinite domain in the time domain using unit triangular force pulses. This matrix is then used during the analysis of the interior finite model to calculate the boundary's response one time step ahead. The approach is exact and does not introduce any approximations. Although the cost of calculating the boundary influence matrix may be substantial, computational savings can be made for situations in which many analyses of the same problem are required.

2. TIME DOMAIN BOUNDARY INFLUENCE MATRIX (TDBIM)

For a linear finite element model, the equation of motion has the general form

$$[M]\{\ddot{u}\} + [K]\{u\} = \{f\} \quad (1)$$

where $[M]$ is the mass matrix; $[K]$ is the stiffness matrix; and $\{f\}$, $\{u\}$, $\{\ddot{u}\}$ are vectors of nodal forces, displacements and acceleration, respectively. For simplicity, damping is not included. Equation (1) can be solved by either explicit or implicit time integration methods, but the proposed artificial boundary treatment is designed for explicit methods. Examples of methods commonly used in wave propagation studies are the well-known explicit central difference method and the implicit Newmark family of methods [3], which includes the central difference method as a special case ($\beta = 0$, $\gamma = 0.5$).

The stiffness matrix $[K]$ and possibly the mass matrix $[M]$ in eqn (1) couple every node in the finite element mesh with its surrounding nodes. Consequently, the true behaviour of the problem is not correctly calculated unless one prescribes the time history of boundary motion (or alternatively boundary reaction forces) which would take place in an extended model (i.e. a very large model in which no reflections occur). In other words, one always needs to know the true response of the boundary nodes one time step ahead of the last time station and prescribe it as a boundary condition to simulate the stiffness contribution of the exterior infinite domain. Because the boundary's response is a function of the wave incidence angle, wave frequency, wave type, etc., the only way to have the true response (i.e. with no reflections) in time domain analyses is to use an extended finite element mesh. The computational cost of this analysis may become substantial if the problem has to be analysed many times. In the solution procedure explained next, an extended model is employed only as a preliminary step to calculate a boundary influence matrix directly in the time do-

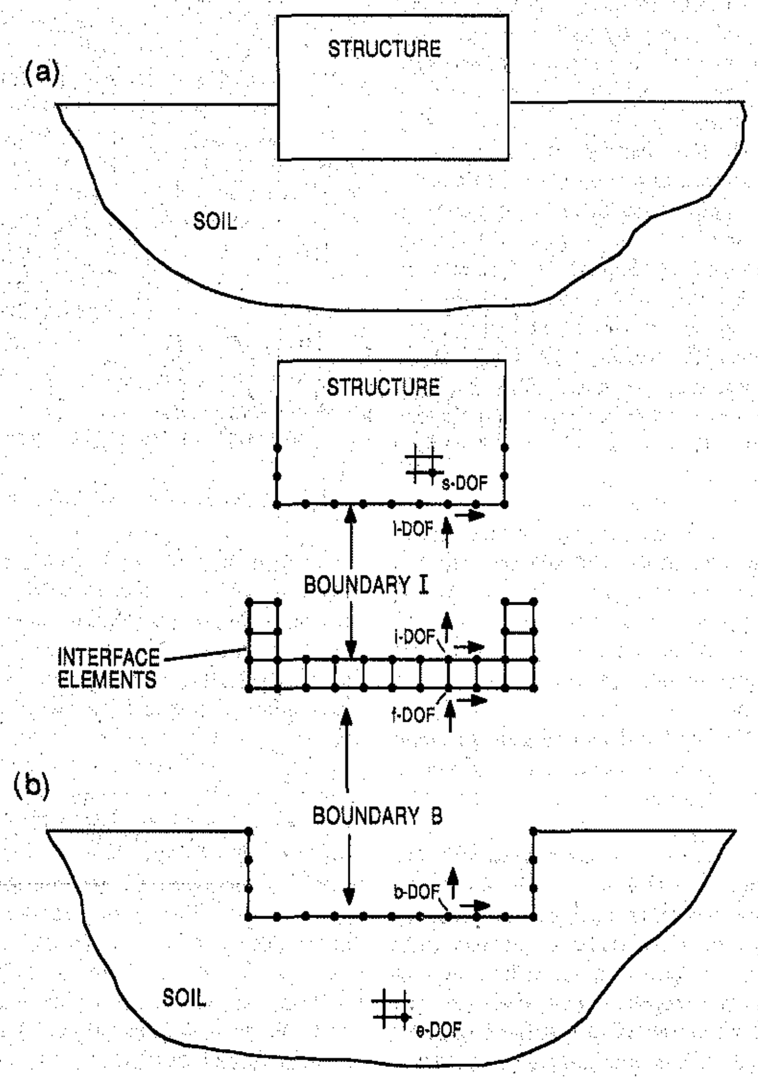

Fig. 1.

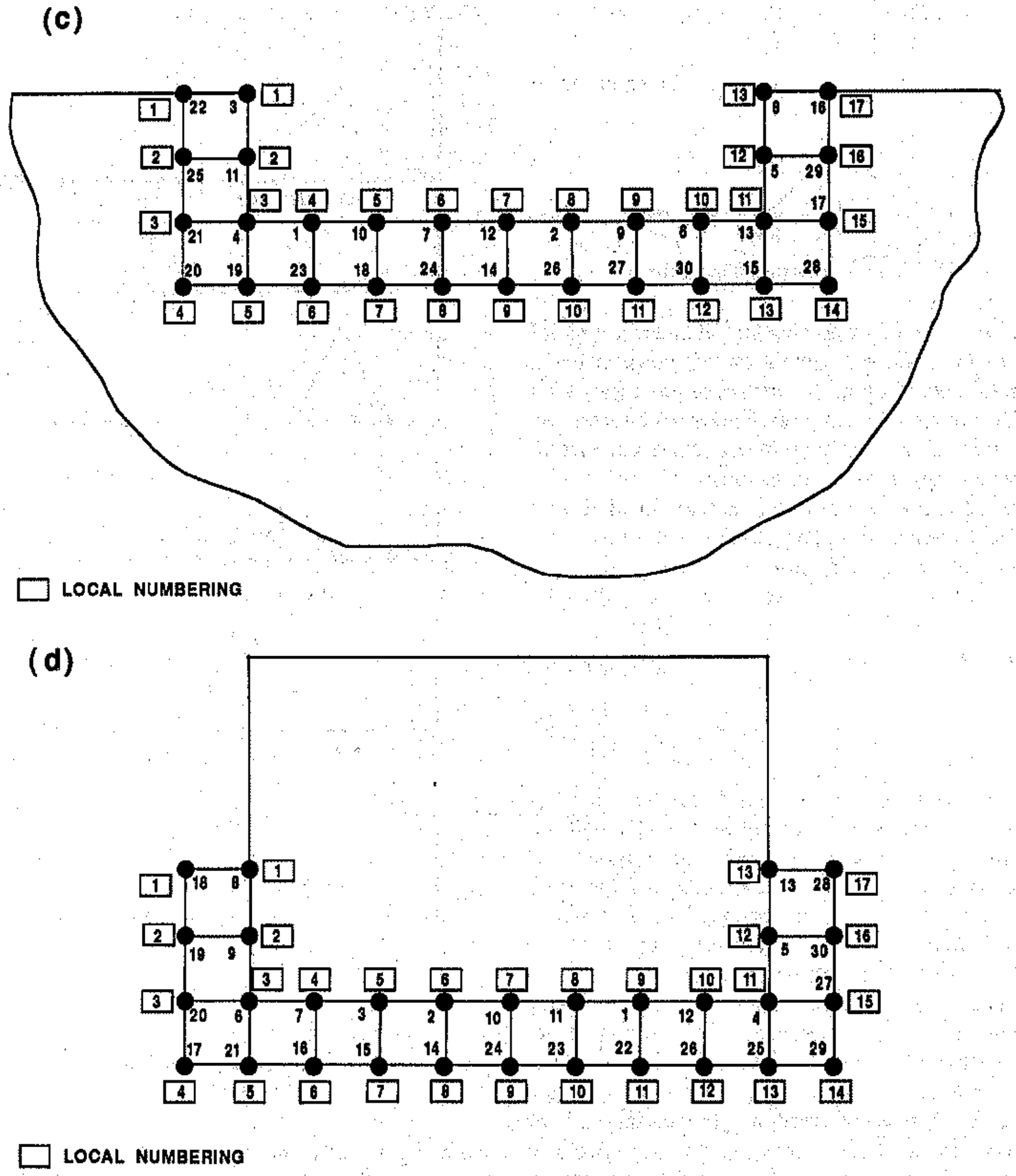

Fig. 1. Substructure of soil–structure system for the proposed solution method.

main. The problem can then be analysed economically using a small finite model as many times as desired.

In the following procedure, the extended finite element model, for example the soil–structure model shown in Fig. 1(a), is substructured into three parts: (1) structure, (2) soil, and (3) interface zone, as shown in Fig. 1(b). The structure part consists of the structure itself and a portion of the supporting soil which may be nonlinear and/or geometrically irregular. The soil part must be linearly elastic (or linearly viscoelastic) and it should be sufficiently large for no reflections at its far boundary to reach the boundary with the structure while the solution is in progress. The interface zone consists of one strip of elements (note: for the sake of simplicity, elements are assumed here to be four-noded) that separate the structure and soil parts and therefore it contains all nodes coupled to degrees of freedom on the boundary **B** of the soil part. The finite element equations of the soil part and interface zone combined may be written in the following form:

$$\begin{bmatrix} m_{ii} & 0 & 0 \\ 0 & m_{ff}+m_{bb} & 0 \\ 0 & 0 & m_{ee} \end{bmatrix} \begin{Bmatrix} \ddot{u}_i \\ \ddot{u}_b \\ \ddot{u}_e \end{Bmatrix} + \begin{bmatrix} k_{ii} & k_{if} & 0 \\ k_{fi} & k_{ff}+k_{bb} & k_{be} \\ 0 & k_{eb} & k_{ee} \end{bmatrix} \begin{Bmatrix} u_i \\ u_b \\ u_e \end{Bmatrix} = \begin{Bmatrix} f_i \\ 0 \\ 0 \end{Bmatrix}. \quad (2)$$

In the above equation, the m and k matrices are submatrices of the mass and stiffness matrices of the soil and interface zone substructures. The mass ma-

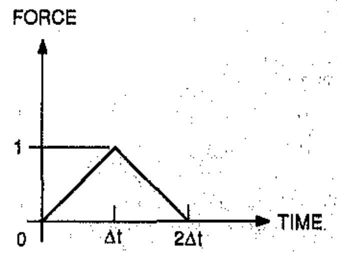

Fig. 2. Unit triangular load pulse.

trices are assumed to be diagonal. $\ddot{u}$ and u are nodal acceleration and displacement vectors, respectively. f_i in eqn (2) and f_i in Fig. 1(b) are the attending internal nodal force vectors acting on boundary **I** between the structure and interface zone. These forces are of equal magnitude but opposite directions.

Similarly, the equations of motion for the finite nonlinear model, consisting of the structure part and interface zone Fig. 1(d), are

$$\begin{bmatrix} m_{ss} & m_{sI} \\ m_{Is} & m_{II}+m_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{u}_s \\ \ddot{u}_i \end{Bmatrix} + \begin{bmatrix} k_{ss} & k_{sI} \\ k_{Is} & k_{II}+k_{ii} \end{bmatrix} \begin{Bmatrix} u_s \\ u_i \end{Bmatrix} = \begin{Bmatrix} f_s \\ -k_{Ib}\cdot u_b \end{Bmatrix}. \quad (3)$$

The structure part can be nonlinear, but for simplicity it is considered linear in the above equation. In eqn (3), it is assumed that the response of boundary **B**, $\{u_b\}$, is known in advance and therefore it is a prescribed boundary condition and therefore the contribution $k_{Ib}\cdot u_b$ is transferred to the right-hand side of the equilibrium equations.

Before solution of eqns (3), a time domain bounary influence matrix $[D]$ is calculated. This matrix describes the response of boundary **B** degrees of freedom resulting from unit triangular pulse forces (see Fig. 2) at degrees of freedom of boundary **I**. For instance, an element (i, j, k) of this matrix is defined as the response of degree of freedom i of boundary **B** at time step k resulting from a unit triangular pulse force at degree of freedom j of boundary **I**. Matrix $[D]$ is calculated by solving eqns (2) for unit triangular pulse forces applied at degrees of freedom of boundary **I**, one at a time. The resulting response at degrees of freedom of boundary **B** is collected in matrix $[D]$. This matrix is then used to calculate the displacements $\{u_b\}$ in eqns (3) as explained next.

As waves propagate from the structure part into the soil part, the attending internal nodal forces $\{f_i\}$ in Fig. 1(d) can be considered to be the source of wave motion of degrees of freedom of boundary **B**, $\{u_b\}$. For the purpose of calculating the boundary response $\{u_b\}$, the actual attending nodal forces $\{f_i\}$ are broken down into triangular pulses as demonstrated in Fig. 3. If an explicit time integration scheme is used (at least for the interface zone), a non-zero displacement response at any of the degrees of freedom on boundary **B** resulting from a triangular pulse,

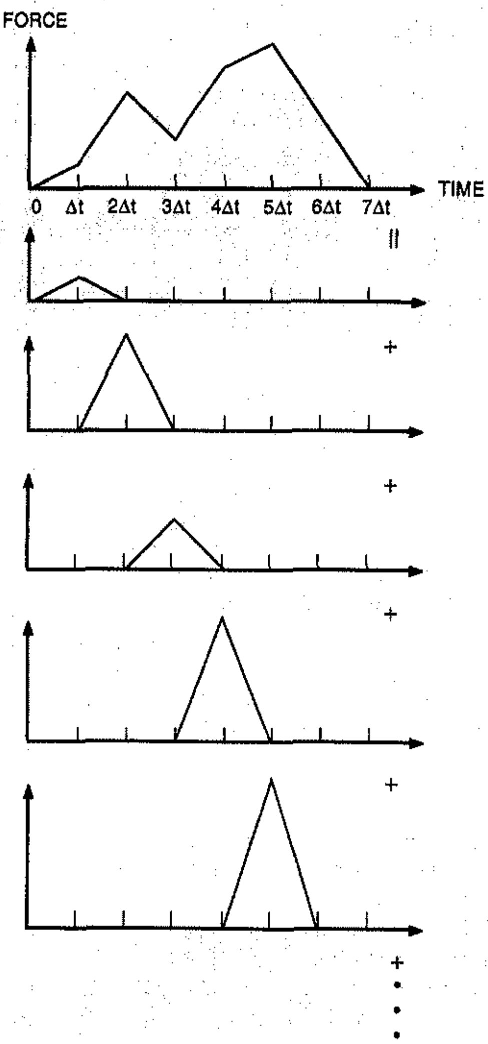

Fig. 3. Representation of force time history by triangular force pulses.

say over time steps m and $(m + 1)$, will start only at the end of time step $(m + 1)$ or later [note: for the central difference time integration scheme it will start at the end of time step $(m + 2)$]. Hence reflections from the boundary nodes to the interior nodes will commence after the end of time step $(m + 1)$. Consequently, the boundary's response may be calculated one time step ahead of the last time station using the attending nodal force vector $\{f_i\}$ at the last and past time stations and the boundary influence matrix $[D]$. The previously determined time domain boundary influence matrix is used to calculate the boundary's responses resulting from the triangular pulse components of the load by simple multiplication. The responses are then superimposed to obtain the boundary's displacement $\{u_b\}$ for the next time step.

3. NUMERICAL IMPLEMENTATION

Numerical implementation of the above solution procedure using the time domain boundary influence matrix (TDBIM) may at first seem cumbersome. In reality, its incorporation into existing finite element codes does not represent additional complications. In the following, a method is used that effectively generates the boundary influence matrix $[D]$ using the soil substructure in Fig. 1(c). The method also effectively calculates the attending nodal vector $\{f_i\}$ and then calculates the boundary displacement vector $\{u_b\}$ which is necessary for the analysis of the structure part in Fig. 1(d).

3.1. *Numbering technique*

The finite element nodes in Fig. 1(c) and (d) of the soil part and the structure, respectively, will usually have global node numbering that does not produce identical node numbers for nodes on the boundary **B** in both models. The same is also true for nodes on the boundary **I**. For nodal points on these boundaries, a local node numbering is defined which coincides in numbers of both models as shown in Fig. 1(c) and (d). The proposed treatment uses two arrays NATB and NCTB. Array NATB has as many elements as the maximum number of nodes on boundary **B**, say NNATB. It stores the global node number corresponding to each node on this boundary. Array NCTB has the same function as array NATB but for nodes on boundary I. Its dimension is NNCTB, which is the maximum number of nodes on boundary **I**. This information is then used to identify these nodes during the calculation of the boundary influence matrix and to extract the appropriate influence coefficients during the calculation of the boundary response.

The boundary influence matrix $[D]$ is calculated using the extended soil model (see Fig. 1c), by applying unit triangular pulses to each degree of freedom of boundary **I**, one at a time. The resulting displacement time histories of degrees of freedom of boundary **B** are appropriately assembled into the matrix $[D]$ as follows: D (LBNEQ, LFENQ, ISTEP) = displacement response at degree of freedom number LBNEQ of boundary **B** at time step ISTEP resulting from unit triangular pulse force at degree of freedom LFNEQ of boundary **I**, whereas all other DOF's on **I** have zero forces (note: LBNEQ and LFNEQ are local numbers for degrees of freedom on boundaries **B** and **I**, respectively).

3.2. *Calculation of internal forces*

The calculation of the boundary's displacement using the time domain boundary influence matrix $[D]$ requires the availability of past time history of the attending nodal forces $\{f_i\}$ at boundary **I**. To calculate this force vector at the end of a time step, the equations of motion of the degrees of freedom for the boundary **I** are used as follows:

$$\{f_i\} = [m_{ii}]\{\ddot{u}_i\} + [k_{ii}]\{u_i\} + [k_{ib}]\{u_b\} \tag{4}$$

where the definitions of vectors and matrices in this equation are the same as for eqn (2). These forces are then stored in an array, called here F, as follows: F(LFNEQ, ISTEP) = attending nodal force acting on degree of freedom number LFNEQ (local number) at the end of time step ISTEP.

3.3. *Calculation of boundary displacement response* $\{u_b\}$

To calculate the boundary displacement response one time step ahead of the last time station ISTEP, the amplitudes of triangular pulses composing the attending nodal force time history $\{f_i\}$ are multiplied by the corresponding influence coefficients of the boundary influence matrix $[D]$ and the results are then superimposed. For instance, the response of degree of freedom LBNEQ at the end of time step (ISTEP + 1) is given by

$$u_{\mathrm{LBNEQ}} = \sum_{i=1}^{\mathrm{LFNEQ}} \sum_{k=1}^{\mathrm{ISTEP}} F(i,k) \cdot D(\mathrm{LBNEQ}, i, \mathrm{ISTEP} - k + 2). \tag{5}$$

These boundary displacements are then prescribed as boundary conditions at the truncation boundary of the finite model in Fig. 1(d).

4. NUMERICAL EXAMPLE

To verify the proposed TDBIM procedure, a plane strain elasticity problem is considered. The problem is that of a layer which is underlain by a rigid base and subjected to a vertical line load at the surface. The finite element model and the corresponding boundary conditions used for this test problem are shown in Fig. 4. The finite element mesh consists of the structure part, interface elements and the soil part as mentioned previously. In this simple example, the structure part comprises only a small soil region which, for simplicity, is considered to be linearly elastic although in general the structure could be nonlinear, as discussed in Section 2. Bilinear isoparametric elements are employed and they are all of the same size. The soil part of the finite element model is sufficiently large for the response of the boundary **B** to be obtained for the duration of the test before any reflected waves arrive from the right boundary.

Material properties of the layer are summarized in Table 1. The surface vertical line load consists of a one-cycle sine pulse. The duration of the cycle is 15 sec and its amplitude is one unit. Time integration is performed with an implicit–explicit algorithm [3]. All elements are considered explicit and the integration parameters employed are $\gamma = 0.5$, $\beta \approx 0.0$, time step $\Delta t = 0.9$ sec. The duration of the analysis is 120 time steps, which is equal to about seven load cycles. No damping is applied. Calculations are performed using double precision on an IBM 3090 computer. The vertical displacement response time history is recorded at point A of Fig. 4.

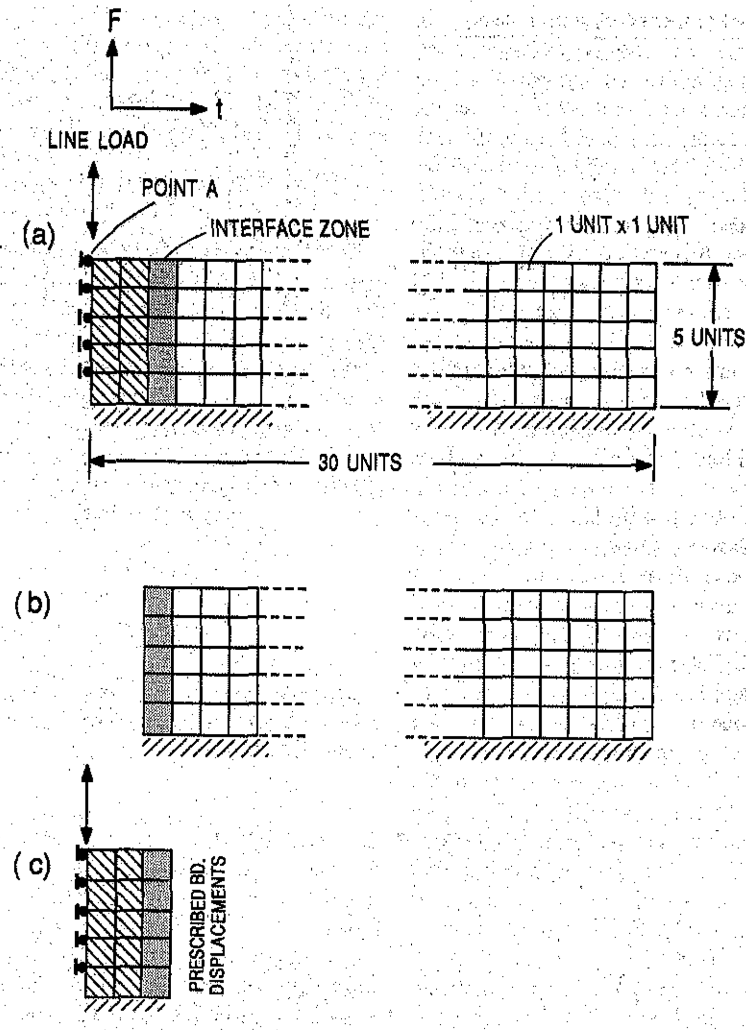

Fig. 4. Finite element model used for surface layer example: (a) complete model; (b) model used for calculation of TDBIM; (c) small finite model (structure).

The finite element model in Fig. 4(b), consisting of the soil part and interface elements, is used to calculate the time domain boundary influence matrix. This matrix is stored and then recalled to calculate the boundary response of the model in Fig. 4(c). Results of the analysis which used the small model in Fig. 4(c) with the TDBIM approach are then compared with the results of the extended mesh in Fig. 4(a), which

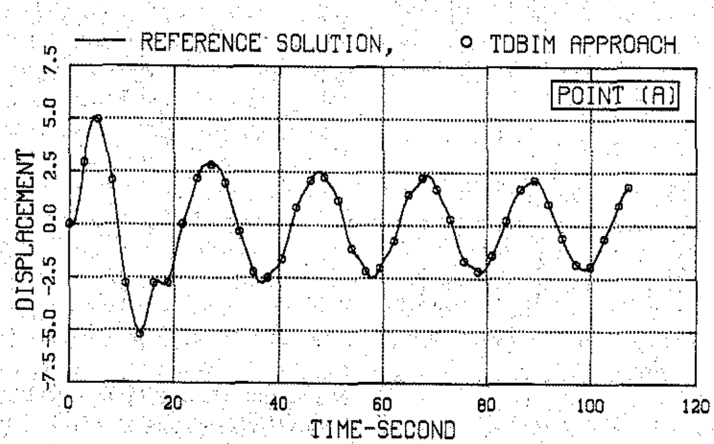

Fig. 5. Vertical displacement response at point A: comparison between reference solution and solution using TDBIM approach.

Table 1. Material properties of layer system

S-wave velocity	0.5345 units/sec
P-wave velocity	1 unit/sec
Poisson's ratio	0.3
Unit density	1.0

are called the reference solution. As expected, the resulting time history at point A for the TDBIM solution coincides with the reference solution. The difference cannot be resolved within the scale of the drawings and it has to be marked by circles as shown in Fig. 5. The time domain boundary influence matrix calculated for this problem may be stored for future use with the same problem, for different loading conditions and/or different material properties of the structure part.

As an extension to this test problem, the efficiency of the popular standard viscous silent boundary [4] is examined. The purpose of this test is to demonstrate the approximation involved when using such local silent boundaries. Two locations of the lateral silent boundary are tested. The first is at a lateral distance equal to one layer depth and the second is at a lateral distance equal to twice the layer's depth, as shown in Fig. 6. Vertical displacement response is recorded at surface points A, B and C. The response obtained by using this viscous boundary is compared with the reference solution which was calculated previously using an extended mesh.

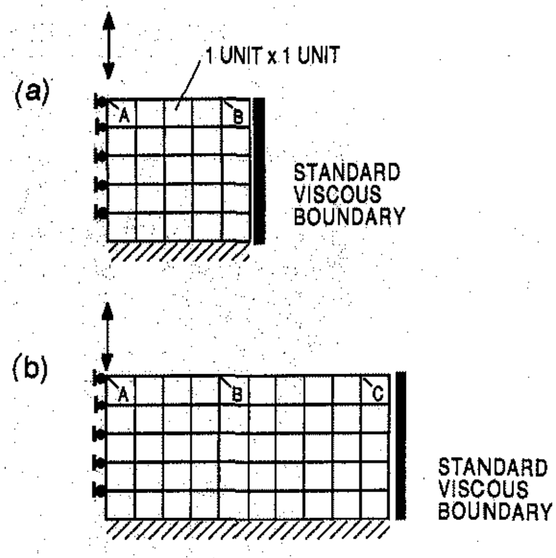

Fig. 6. Finite element models using standard viscous boundary: (a) small model; (b) large model.

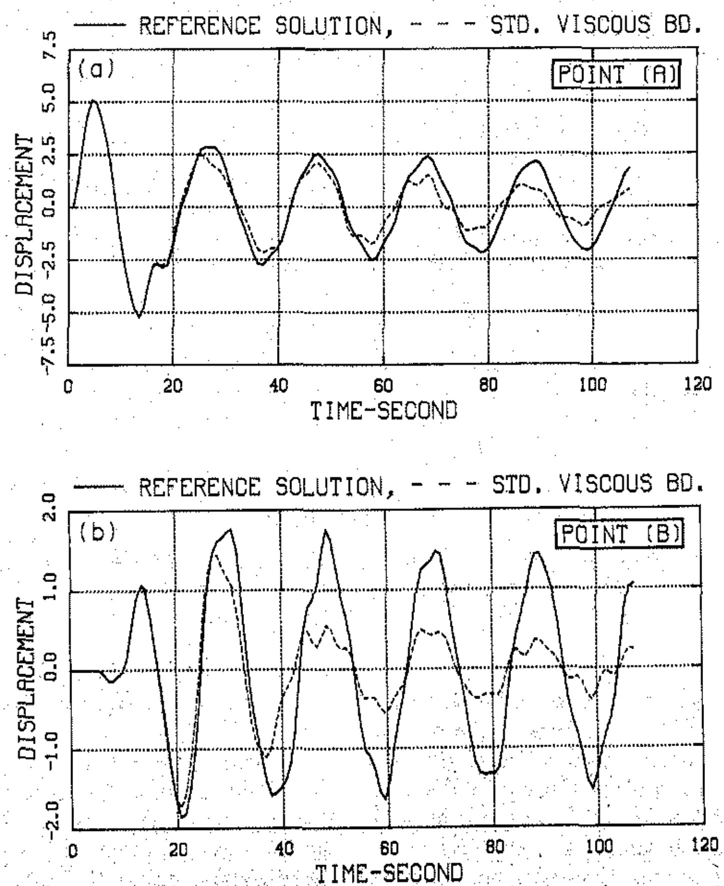

Fig. 7. Vertical displacement response: comparison between reference solution and small model employing viscous boundary (a) at point A, (b) at point B.

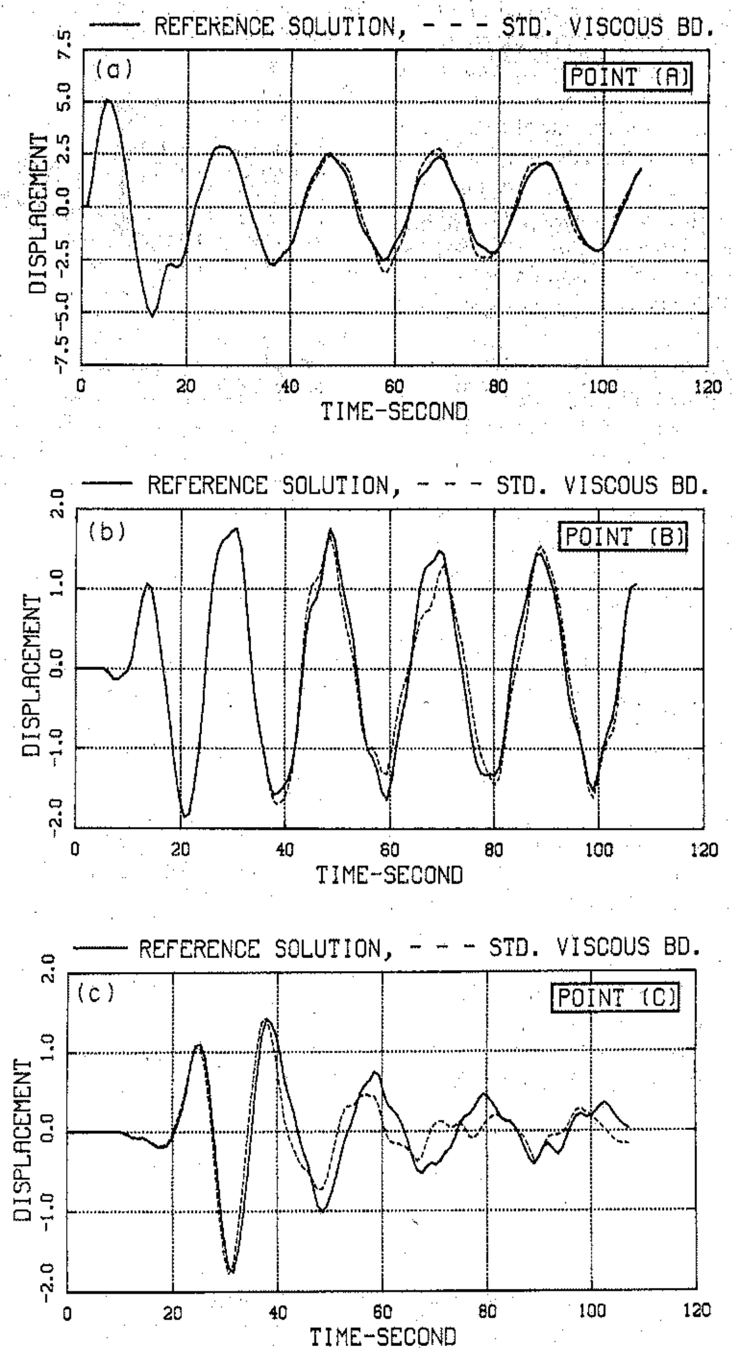

Fig. 8. Vertical displacement response: comparison between reference solution and large model employing viscous boundary (a) at point A, (b) at point B, (c) at point (C).

Results of the small model, shown in Fig. 7, are not in good agreement with the reference solution. Results of the larger model, shown in Fig. 8, represent substantial improvement over the results of the small model for points A and B but the response at point C located beside the boundary is poor. This means that for the viscous boundary, more accurate answers at point C require that the boundary be moved further away.

5. SUMMARY

An economical and accurate method is presented for the analysis of problems involving wave propagation in unbounded media. The method uses an influence matrix, calculated in the time domain, for the boundary between the finite model of interest and the truncated unbounded domain. During analysis of the finite model, this matrix is used to calculate the boundary's response one time step ahead. This response is then used to simulate the stiffness contribution of the truncated unbounded domain and hence preserve the real physical behaviour of the problem. The advantage of this procedure is that the boundary matrix can be used for as many analyses as desired of the finite model at minimal additional computational cost. The load conditions, geometry and material properties of the finite model are arbitrary.

In the numerical sense, the formulation of this method and the results obtained are exact.

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REFERENCES

1. J. E. Luco, A. H. Hadjian and H. D. Bos, The dynamic modeling of the half-plane by finite elements. *Nucl. Engng Des.* **31**, 184–194 (1974).
2. E. Kausel, Local transmitting boundaries. *J. Engng Mech.* **114**, 1011–1027 (1988).
3. T. J. R. Hughes and W. K. Liu, Implicit–explicit finite elements in transient analysis: implementation and numerical examples. *J. appl. Mech.* **45**, 375–378 (1978).
4. J. Lysmer and R. L. Kuhlemeyer, Finite dynamic model for infinite media. *J. Engng Mech. Div., ASCE* **95**, 859–877 (1969).